DYNAMICS OF THE PHASE-FIELD SYSTEMS

BY

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To God Almighty and to my entire family

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THESIS ABSTRACT

NAME: CYRIL DENNIS ENYI

TITLE OF STUDY: Dynamics of the phase-field systems

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We investigate three models of Phase-field systems of phase transition, coupling temperature with a continuous order parameter which describe degree of solidification in material sciences. We show well posedness, existence of the global attractor, exponential attractors and inertial manifolds. These objects describe the large time behaviour of dynamical systems. Considering some singular parameters in the equations, we examine continuity properties of the attractors and inertial manifolds. This shows the convergence of the dynamics to those of the Cahn-Hilliard equations and the Caginalp phase-field system.

ملخص الرسالة

الإسم: سيريل دنيس أنيي

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نقوم ببحث ثلاثة نماذج من أنظمة مناطق الطورللطورات الانتقالية التي تقرن درجة الحرارة مع الوسائط ذات الترتيب المستمر والتي تصف درجة التصلب في العلوم المادية. سوف نبين الصياغة الجيدة، ووجود جاذب شامل، وجاذب الأسي، ومنطو عطالة. إن هذه المفاهيم تصف السلوك طويل المدى للنظم الديناميكية. وباعتبار بعض الوسائط المفردة لبعض المعادلات، نفحص خصائص الاستمرارية للجاذبين ومنطو عطالة. وهذا يدل على تقارب الديناميكيات إلى تلك المتعلقة بمعادلات كاهن-هيليارد وأنظمة مناطق الطور لكاجينالب

CHAPTER 1

INTRODUCTION

The global attractor.

The *global attractor* $\mathcal{A}$ is a compact invariant set lying on the phase space which attracts uniformly the trajectories starting from bounded sets when time goes to infinity (see, e.g., [21, 68, 71]). However, the global attractor may have a complicated fractal structure, even for finite-dimensional dynamical systems, and a reasonably explicit description of the dynamics on the attractor might be out of reach.

Given that $u(t) \in E$ an infinte-dim phase space, $\exists\, t_0$ such that $u(t) \in \mathcal{A}$, $\forall\, t \geq t_0$.

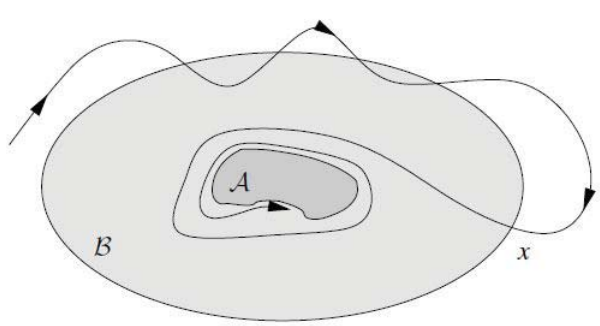

Figure 1.1: Attractor (cf. [52, pg. 17])

In case the fractal dimension of $\mathcal{A}$, i.e., $\dim_F \mathcal{A} = \limsup_{\varsigma \to 0^+} \frac{\ln \mathrm{N}(\varsigma, \mathcal{A})}{\ln(1/\varsigma)} < \infty$, (cf. Definition 1.6).

☞ **Question**: Is there a system of ODE whose solution behave like $u(t)$ for large t?

Inertial manifolds.

An *inertial manifold* is a positively invariant smooth finite-dimensional manifold which contains the global attractor and attracts trajectories at a uniform exponential rate (see [23, 52, 68, 71]). It follows that the *a priori* infinite-dimensional dynamical system reduces, on the inertial manifold, to a finite system of ordinary differential equations.

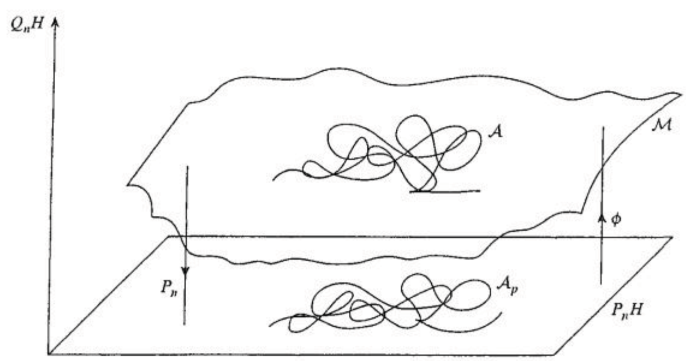

Figure 1.2: Inertial Manifold $\mathcal{M}$, $P_nH \simeq \mathbb{R}^n$. (cf. [68, pg. 386])

$u(t) \in \mathcal{A} \;\Rightarrow\; u(t) = p(t) + \Phi(p(t))$, $p(t) \in \mathbb{R}^n$ is solution of an ODE.

Most of the current methods for construction of inertial manifolds use the *spectral gap condition*, which requires a sufficient gap in the spectrum of the linear operator associated with the PDE. This condition does not hold in general for many dynamical systems. In this case, it is still possible to construct an intermediate object between the global attractor and the inertial manifold which is called *exponential attractor*.

Exponential attractors.

This set is a compact and positively invariant set with finite fractal dimension which attracts all the trajectories starting from bounded sets at a uniform exponential rate (see [1, 26, 30, 32, 41, 53]). The sensitivity of the global attractor, exponential attractors and inertial manifolds under small perturbations is the main focus in this thesis. One may see [58] for some recent developments in the construction of exponential attractors, and [75] for a recent survey on inertial manifolds and finite-dimensional reduction for dissipative PDEs.

The main purpose of this thesis is to investigate the well-posedness, the existence and continuity properties of the global attractor, exponential attractors and inertial manifolds for three models of the phase-field systems. In Chapters 2 and 3, we show convergence of the dynamics of a conserved phase-field system to those of the viscous and non-viscous Cahn-Hilliard equations respectively, as the heat capacity and viscosity parameters approach zero. In Chapter 4, we considered a conserved phase-field equation based on the theory of heat conduction involving two temperatures, and study the convergence of the dynamics to those of the conserved phase-field equation studied in Chapter 2. In the last Chapter 5, we prove the existence of a robust family of exponential attractors for a parabolic-hyperbolic phase-field system.

1.1 Main results

Throughout this thesis, Ω is either a bounded domain of $\mathbb{R}^d$ with smooth boundary, or $\Omega = \Pi_{i=1}^d (0, L_i)$, $L_i > 0$, for $d \leq 3$. Firstly, we consider the following conserved phase-field system

$$\tau\phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi) - u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$\varepsilon u_t + \phi_t - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+.$$

In the first case, for $\delta > 0$ arbitrarily fixed, we prove a well posedness result, the existence of the global attractor, exponential attractors and inertial manifolds. Then we show the convergence of the dynamics to those of the viscous Cahn-Hilliard equation as $\varepsilon = 0$. Precisely, we show the upper semicontinuity of the global attractor, robustness of the exponential attractors and inertial manifolds at $\varepsilon = 0$ in the sense of upper and lower semicontinuity. In the second case, we show convergence results with respect to both the parameters (ε, δ) as this pair tends to $(0, 0)$, thus establishing convergence of the dynamics of the system to those of the Cahn-Hilliard equation.

Secondly, we consider the conserved phase-field system which is based on the theory of heat conduction involving two temperatures, namely

$$\tau\phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi) - u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$\sigma u_t - \epsilon\Delta u_t + \phi_t - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+.$$

We prove a well posedness result, the existence of the global attractor, exponential attractors and inertial manifolds. Then we show the convergence of the dynamics to those of the Calginalp phase-field system as $\epsilon \to 0$. Precisely, we show the upper semicontinuity of the global attractor, robustness of the exponential attractors and inertial manifolds at $\epsilon = 0$.

Lastly, we consider the following parabolic-hyperbolic phase-field system

$$\epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi) - u = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$u_t + \phi_t - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+.$$

We construct a family of robust exponential attractors which are both upper and lower semicontinuous at $\epsilon = 0$. Hence obtaining the convergence of the dynamics of this system to those of the the Caginalp phase-field system.

1.2 Models description

The phase-field system is a parabolic system of equations describing the temperature u and a "phase-field" known as order parameter ϕ. The temperature u is scaled such that $u = 0$ represents the ordinary planar melting equilibrium temperature, and ϕ is scaled such that ϕ close to $+1$ represents the liquid phase while ϕ near -1 represents the solid phase. The interface therefore is defined to be the set of all points at which $\phi = 0$, i.e., where ϕ vanishes (see [9, 12] for more details).

In this thesis, we shall consider and analyse different models of the phase-field systems, two of which are of the conserved type, while the third model is of the non-conserved type (cf. [10, 15]).

- **Model 1.** For an isotropic material, a conserved order parameter ϕ may be expected to satisfy (cf. [12, 13, 14, 64], see also [43, 60]) the constitutive law;

$$\tau\phi_t = \Delta\left(\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}\phi} + \delta\phi_t\right), \tag{1.2.1}$$

where $\delta \geq 0$ is a viscosity parameter, $\tau > 0$ is a relaxation time and

$$\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}\phi} = -\Delta\phi + g(\phi) - u$$

is the variational or functional derivative of the free energy

$$\mathcal{F}(\phi, u) = \int_\Omega \Big(\frac{1}{2}|\nabla\phi|^2 + G(\phi) - \phi u\Big)dx. \tag{1.2.2}$$

G is a double-well potential and $G' = g$, $G(\phi) = \dfrac{1}{4}(\phi^2 - 1)^2$ is a prototype example which is common to many models in statistical mechanics and quantum field theory.

Now, we assume the classical Fourier law

$$q = -\frac{1}{\varepsilon}\nabla u, \tag{1.2.3}$$

and the constitutive law

$$H_t = -\operatorname{div} q, \tag{1.2.4}$$

where $\varepsilon > 0$ is the heat capacity, q is the thermal flux vector and $H = u + \phi$ is the enthalpy.

From (1.2.1)-(1.2.4), we have the conserved phase-field system (see [13, 14, 43, 60])

$$\tau\phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi) - u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.2.5}$$

$$\varepsilon u_t + \phi_t - \Delta u = 0, \qquad \text{in } \Omega \times \mathbb{R}^+. \tag{1.2.6}$$

When $\varepsilon = 0$, then (1.2.5)-(1.2.6) reduces to the viscous Cahn-Hilliard equation for the single unknown, namely

$$(1+\tau)\phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi)) = 0, \qquad (1.2.7)$$

(cf. [35, 40, 64]), see also [2, 17, 34].

Likewise, when $\varepsilon = \delta = 0$ in (1.2.5)-(1.2.6) we obtain the Cahn-Hilliard equation (cf. [16, 66])

$$(1+\tau)\phi_t - \Delta(-\Delta\phi + g(\phi)) = 0. \qquad (1.2.8)$$

Both (1.2.7) and (1.2.8) are very important in material sciences, see [25, 36, 77].

- **Model 2.**

Gurtin and Williams [48, 49] suggested that there were no a priori grounds for assuming that the second law of thermodynamics for non simple materials involve only a single temperature. They further suggested that it is more reasonable to consider a second law in which the entropy contribution due to heat conduction was governed by one temperature, and that of the heat supply by another temperature. Based on this theory of two temperatures, in [18, 50] (see also [19, 20, 74]) Chen and Gurtin showed that for an isotropic body the linearised constitutive equations for the heat flux $\mathbf{q}$ and thermodynamic temperature θ are respectively

$$\mathbf{q} = -\nabla u \quad \text{and} \quad \theta = \sigma u - \epsilon\Delta u, \qquad (1.2.9)$$

where u is the conductive temperature, $\sigma > 0$ is the heat capacity and $\epsilon > 0$ is a scalar called the temperature discrepancy. With $\mathrm{div}\mathbf{q}$ been the scalar heat supply, from (1.2.9), we obtain

$$\mathrm{div}\mathbf{q} = -\Delta u \quad \text{and} \quad \theta - \sigma u = \epsilon\,\mathrm{div}\mathbf{q}. \qquad (1.2.10)$$

For isotropic materials, it was further suggested that the enthalpy H in the constitutive law (1.2.4) be replaced by $H = \theta + \frac{1}{2}\phi$. Thus, giving rise to

$$\sigma u_t - \epsilon \Delta u_t + \phi_t - \Delta u = 0. \tag{1.2.11}$$

Coupling (1.2.5) and (1.2.11), we obtain the following conserved phase-field system (see [3] for a variant model);

$$\tau \phi_t - \Delta(\delta \phi_t - \Delta \phi + g(\phi) - u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.2.12}$$

$$\sigma u_t - \epsilon \Delta u_t + \phi_t - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+. \tag{1.2.13}$$

- **Model 3.**

Due to the response of ϕ to a generalized force $\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}\phi}$, it was proposed in [44] that ϕ be subjected to a delay expressed by a time dependent relaxation kernel "k", thus we have the relation

$$\phi_t = -\int_{-\infty}^{t} k(t-s) \frac{\mathrm{D}\mathcal{F}}{\mathrm{D}\phi}(s) ds, \tag{1.2.14}$$

for an appropriate relaxation kernel k. An easy but classical choice of the kernel is

$$k(t) = \frac{1}{\epsilon} e^{-\frac{t}{\epsilon}},$$

where $\epsilon > 0$ is a small parameter. When we differentiate (1.2.14) with respect to t, we find

$$\phi_{tt} = -\frac{1}{\epsilon}\phi_t - \frac{1}{\epsilon}\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}\phi},$$

which leads to the parabolic-hyperbolic phase-field system (see [45])

$$\epsilon \phi_{tt} + \phi_t - \Delta \phi + \phi + g(\phi) - u = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.2.15}$$

$$u_t + \phi_t - \Delta u = 0, \quad \text{in } \Omega \times \mathbb{R}^+. \tag{1.2.16}$$

When $\epsilon \to 0$, then $k(t) \to \delta_0(t)$ ($\delta_0(t)$ is the Dirac mass at 0) and (1.2.14) converges to the Caginalp phase-field system:

$$\begin{cases} \phi_t - \Delta\phi + \phi + g(\phi) - u = 0, \\ u_t + \phi_t - \Delta u = 0. \end{cases} \tag{1.2.17}$$

1.3 Literature review

Gilardi [43] proved a well-posedness result for a problem in 3d including Problem (1.2.5)-(1.2.6), with irregular potentials such as logarithmic functions, and subject to dynamic boundary conditions. Global and exponential attractors and also some stability results were proven in [59, 60] for a conserved phase-field system with viscosity and memory terms in 3D, while subjected to Neumann boundary conditions.

Global and exponential attractors were proven by Miranville [54] for Problem (1.2.5)-(1.2.6) with $\delta = 0$ and $\varepsilon = 1$. There, he considered a large class of potential containing polynomials of arbitrary even degree with positive leading coefficients, improving previous results of [8, 11]. The problem (1.2.5)-(1.2.6) with $\delta = 0$, subject to Neumann boundary conditions, was also considered by Bonfoh [5] recently. There he proved the existence of the global attractor and constructed a robust family of exponential attractors. He also proved the existence of inertial manifolds in one space dimension, and for the case of a rectangular domain in two space dimensions. Continuity properties of the intersection of the inertial manifolds with bounded absorbing sets at $\varepsilon = 0$ were also proven. In this thesis, we will follow the method of [5]. Models with dynamic boundary conditions were investigated in [38, 39, 42]. Finally, we also mention a few earlier works [4, 27, 28, 29, 38, 39, 42, 57] on non-conserved phase-field systems where convergence to the Cahn-Hilliard equations were proven.

Bangola [3], considered a non-conserved phase-field system

$$\phi_t - \Delta\phi + f(\phi) = u - \Delta u,$$

$$u_t - \Delta u_t + \phi_t - \Delta u = 0,$$

based on the theory of two temperatures, with the heat flux $\mathbf{q}$ obeying the classical Fourier law (1.2.3) but with a free energy slightly different from (1.2.2). Considering Neuman boundary conditions, and in space dimensions 2 and 3, he proved the existence of the global attractor and exponential attractors. Miranville and Quintanilla [55] considered a conserved phase-field system

$$\begin{cases} \phi_t - \Delta(-\Delta\phi + \phi + f(\phi) - u) = 0, \\ (I - \Delta)(u_{tt} + u_t) - \Delta u = -\Delta\phi, \end{cases}$$

which is based on the theory of two temperatures, however the heat flux $\mathbf{q}$ was assumed to obey the Maxwel-Cattaneo law instead of the classical Fourier law (1.2.3). Subject to both Dirichlet and Neuman boundary conditions, they proved a well posedness result and dissipativity of the associated semigroup of operators. Miranville and Quintanilla [56], studied a phase-field system

$$\begin{cases} \phi_t - \Delta\phi + f(\phi) = u_t - \Delta u_t, \\ (I - \Delta)u_{tt} + (I - \Delta)\phi_t - \Delta u = 0, \end{cases}$$

which is based on the theory of type III thermomechanics with two temperatures for the heat conduction. They proved a well-posedness result and established dissipativity of the associated semigroup of operators.

Grasselli and Pata [44] showed a well-posedness result and the existence of the global attractor for the system ($\mu > 0$)

$$\begin{cases} \mu\phi_{tt} + \phi_t - \Delta\phi + \phi^3 = \gamma(\phi) + \lambda'(\phi)u, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = f. \end{cases}$$

Grasselli and Pata [45] considered the system ($\mu > 0$)

$$\begin{cases} \mu\phi_{tt} + \phi_t - \Delta\phi + \phi - \lambda'(\phi)u + h(\phi) = \xi, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = 0, \end{cases} \tag{1.3.1}$$

in 3D, subject to mixed boundary conditions, Neumann on ϕ and Dirichlet on u. They proved a well-posedness result, the existence of the global attractor and its upper semicontinuity at $\mu = 0$, and constructed exponential attractors. Also, Grasselli et.al. [46] gave a well-posedness result and constructed a robust family of exponential attractors $\mathbb{E}_\mu$ for the system

$$\begin{cases} \mu\phi_{tt} + \phi_t - \Delta\phi - \lambda'(\phi)u + \chi(\phi) = \xi, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = 0. \end{cases} \tag{1.3.2}$$

in 3D, subject to Dirichlet boundary condition, where $\chi(\phi)$ is singular at $\phi = \pm 1$ e.g., $\ln\left(\dfrac{1+\phi}{1-\phi}\right)$, $\phi \in (0,1)$. More precisely, they showed that there exist $c > 0$ and $\varpi \in (0,1)$ both independent of μ such that

$$\operatorname{dist}_K^{\text{sym}}(\mathbb{E}_\mu, \mathbb{E}_0) \leq c\mu^{\varpi}, \quad \forall \mu \in [0,1],$$

in the norm $\|(\phi, \phi_t, u)\|_K^2 = \|\Delta\phi\|_{L^2(\Omega)}^2 + \mu\|\nabla\phi_t\|_{L^2(\Omega)}^2 + \|\phi_t\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2$.

Finally, we would also like to mention the papers [47, 72, 73] where the convergence to equilibrium of solutions for a parabolic-hyperbolic phase-field model were proven.

1.4 Useful lemmas and definitions

Let E be a Banach space.

Definition 1.1 *The family of nonlinear operators* $S(t): \ E \to E, \quad$ *for all* $t \geqslant 0$, *defines a semigroup, if the following conditions are satisfied*

(i) $S(0) = I$;

(ii) $S(t+s) = S(s) \circ S(t) = S(t) \circ S(s)$;

(iii) $S(t) : E \to E$ *is continuous for every* $t \geqslant 0$.

Definition 1.2 *A set* $\mathscr{B} \subset E$ *is an absorbing set for the semigroup* $S(t) : E \to E$ *if given any bounded set* $\mathcal{B} \subset E$ *there exist a time* $t_0(\mathcal{B})$ *such that* $S(t)\mathcal{B} \subset \mathscr{B}$, *for every* $t \geq t_0(\mathcal{B})$.

Definition 1.3 *The semigroup* $\{S(t)\}_{t\geqslant 0}$ *is uniformly compact in* E, *if for any bounded set* $B \subset E$, *there exists* $t(B)$ *such that* $\overline{\bigcup_{t\geqslant t(B)} S(t)B}$ *is compact in* E.

Definition 1.4 *The* $\omega-$*limit of a bounded set* $B \subset E$ *is defined by* $\omega(B) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)B}$.

The Hausdorff semi-distance with respect to the metric of E is defined as:

$$\mathrm{dist}_E(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_E,$$

whereas the symmetric Hausdorff distance between A and B is

$$\mathrm{dist}_E^{\mathrm{sym}}(A, B) = \max \{\mathrm{dist}_E(A, B), \mathrm{dist}_E(B, A)\}.$$

We have the following formal definition of the global attractor (cf. [68, 71]).

Definition 1.5 (The global attractor) *The global attractor for the semigroup* $\{S(t)\}_{t\geqslant 0}$ *is a compact set* $\mathcal{A} \subset E$ *such that*

(i) $\mathcal{A}$ *is invariant, that is,* $S(t)\mathcal{A} = \mathcal{A}, \quad \forall \quad t \geqslant 0$*;*

(ii) $\mathcal{A}$ *attracts all bounded sets, that is, for all* B *bounded in* E*,*

$$\lim_{t\to\infty} dist_E(S(t)B, \mathcal{A}) = 0.$$

The following theorem guarantees the existence of a the global attractor.

Theorem 1.1 ***[71, Chap. 1, Theorem 1.1]*** *Let* $\{S(t)\}$ *be compact (or) uniformly compact (or) asymptotically compact (or)* $S(t)$ *has a splitting* $S(t) = S_1(t) + S_2(t)$*, where* $S_1(t)$ *is uniformly compact while* $S_2(t) : E \to E$ *is continuous for each* $t \geqslant 0$ *and* $\lim_{t\to\infty} \sup_{u\in B} \|S_2(t)u\|_E = 0, \ \forall B \subset E$ *bounded. We assume that there exists a bounded absorbing set* B_0 *in* E*. Then,* $\mathcal{A} = \omega(B_0)$ *is the compact global attractor. Furthermore, if* E *is connected, then* $\mathcal{A}$ *is also connected.*

Definition 1.6 *[cf. [58]] Let* $X \subset E$ *ba a (relatively) compact set. The fractal dimension of* X *is defined by*

$$\dim_{\mathrm{F}} X = \limsup_{\varsigma\to 0^+} \frac{\ln \mathrm{N}(\varsigma, X)}{\ln(1/\varsigma)},$$

where $\varsigma > 0$ *and* $\mathrm{N}(\varsigma, X)$ *is the minimal number of balls in* E *of radius* ς *which are necessary to cover* X*.*

Definition 1.7 (Exponential attractor) *Let* X *be a compact subset of* E*. A compact set* $\mathcal{E} \subset X$ *is called an exponential attractor for the semigroup* $S(t)$ *for the topology of* E *if:*

(i) $\mathcal{E}$ *is positively invariant under* $S(t)$*, that is,* $S(t)\mathcal{E} \subset \mathcal{E}, \quad \forall t \geq 0$;

(ii) *the fractal dimension of* $\mathcal{E}$ *is finite;*

(iii) *there exists a constant* $c_0 > 0$ *such that, for every bounded subset* $B \subset X$, *there exists a constant* $c_1(B) > 0$ *such that*

$$\operatorname{dist}_E(S(t)B, \mathcal{E}) \le c_1 e^{-c_0 t}, \ \ \forall t \ge 0.$$

The following theorem gives sufficient conditions ensuring the existence of a robust family of exponential attractors (see [6, 41]).

Theorem 1.2 ***([6, 41])*** *Let* E^1, E^2, V^1, V^2, W^1, W^2 *be Banach spaces such that the embeddings* $W^i \hookrightarrow V^i \hookrightarrow E^i$, $i = 1, 2$, *are compact. Set* $E_\varepsilon = E^1 \times E^2$, $V_\varepsilon = V^1 \times V^2$, $W_\varepsilon = W^1 \times W^2$, *with the convention that* $E_0 = E^1$, $V_0 = V^1$, *and* $W_0 = W^1$. *We endow* $X = X_1 \times X_2$ *where* X_1, X_2 *are Banach spaces, with the following norm*

$$\|(p,q)\|_X = (\|p\|^2_{X_1} + \varepsilon\|q\|^2_{X_2})^{1/2}.$$

Let $B_\varepsilon(r)$ *denote a closed ball in* W_ε *of radius* $r > 0$ *and centered at zero. Consider a one-parameter family of strongly continuous semigroups* $\{S_{\varepsilon,\delta}(t)\}_{\varepsilon,\delta}$ *acting on the phase-space* E_ε, *for each* $\varepsilon \in [0,1]$ *and* $\delta \in [0,1]$. *Then assume that there exist* $\alpha, \beta, \gamma, \vartheta \in (0,1]$, $\kappa \in (0, \frac{1}{2})$, $\Upsilon_j \ge 0$, *and* $\varrho > 0$ *(all independent of* ε *and* δ*) such that, setting* $B_\varepsilon = B_\varepsilon(\varrho)$, *the following conditions hold:*

1. *There exists a map* $\mathscr{L}_\delta : B_0 \to V^2$ *which is Hölder continuous of exponent* α, *uniformly with respect to* δ. *Here* B_0 *is endowed with the metric topology of* E^1.
2. *There exists* $t^\star > 0$, *independent of* ε *and* δ *such that*

$$S_{\varepsilon,\delta}(t)B_\varepsilon \subset B_\varepsilon, \ \ \forall t \ge t^\star,$$

and B_ε *is uniformly bounded (with respect to* ε *and* δ*) in the* E_1-*norm. Moreover, setting* $S_{\varepsilon,\delta}(t^\star) = S_{\varepsilon,\delta}$, *the map* $S_{\varepsilon,\delta}$ *satisfies, for every* $z_1, z_2 \in B_\varepsilon$,

$$S_{\varepsilon,\delta}z_1 - S_{\varepsilon,\delta}z_2 = L_{\varepsilon,\delta}(z_1, z_2) + K_{\varepsilon,\delta}(z_1, z_2),$$

$$\text{where} \quad \|L_{\varepsilon,\delta}z_1 - L_{\varepsilon,\delta}z_2\|_{E_\varepsilon} \leq \kappa\|z_1 - z_2\|_{E_\varepsilon},$$

$$\text{and} \quad \|K_{\varepsilon,\delta}z_1 - K_{\varepsilon,\delta}z_2\|_{V_\varepsilon} \leq \Upsilon_1\|z_1 - z_2\|_{E_\varepsilon}.$$

3. *For any $z \in B_\varepsilon$, there hold*

$$\|S^m_{\varepsilon,\delta}z - \mathcal{L}_{\varepsilon,\delta}S^m_{0,0}\Pi_\varepsilon z\|_{E_1} \leq \Upsilon_2^m(\varepsilon + \delta)^\beta, \quad \forall m \in \mathbb{N}, \tag{1.4.1}$$

$$\|S_{\varepsilon,\delta}(t)z - \mathcal{L}_{\varepsilon,\delta}S_{0,0}(t)\Pi_\varepsilon z\|_{E_1} \leq \Upsilon_3(\varepsilon + \delta)^\gamma, \quad \forall t \in [t^\star, 2t^\star]. \tag{1.4.2}$$

Here the "lifting" map $\mathcal{L}_{\varepsilon,\delta} : B_0 \to E_\varepsilon$ is defined by

$$\mathcal{L}_{\varepsilon,\delta}x = \begin{cases} (x, \mathscr{L}_\delta x), & \text{if } \varepsilon > 0, \\ x, & \text{if } \varepsilon = 0, \end{cases}$$

and $\Pi_\varepsilon : B_\varepsilon \to B_0$ is the projection onto the first component when $\varepsilon > 0$, and the identity map otherwise.

4. *The map $z \mapsto S_{\varepsilon,\delta}(t)z$ is Lipschitz continuous on B_ε endowed with the metric topology of E_ε, with a Lipschitz constant independent of ε, δ and $t \in [t^\star, 2t^\star]$.*

5. *The map*

$$(t, z) \mapsto S_{\varepsilon,\delta}(t)z : [t^\star, 2t^\star] \times B_\varepsilon \to B_\varepsilon$$

is Hölder continuous of exponent ϑ (we do not require uniformity with respect to ε and δ), where B_ε is endowed with the metric topology of E_ε.

Then there exists a family of exponential attractors $\mathcal{E}_{\varepsilon,\delta}$ on $\mathcal{B}_\varepsilon = \overline{B_\varepsilon}^{E_\varepsilon}$ with the following properties:

(i) *$\mathcal{E}_{\varepsilon,\delta}$ attracts $\mathcal{B}_\varepsilon$ with an exponential rate which is uniform with respect to ε*

and δ, that is,

$$\mathrm{dist}_{E_\varepsilon}(S_{\varepsilon,\delta}(t)\mathcal{B}_\varepsilon, \mathcal{E}_{\varepsilon,\delta}) \leq M_1 e^{-\omega t}, \quad \forall t \geq 0,$$

for some $M_1 > 0$ and some $\omega > 0$.

(ii) *The fractal dimension of $\mathcal{E}_{\varepsilon,\delta}$ is uniformly bounded with respect to ε and δ, that is,*

$$\dim_F(\mathcal{E}_{\varepsilon,\delta}) \leq \frac{\ln \mathrm{N}\left(\dfrac{\varsigma}{2\Upsilon_1}, \mathrm{B}_1^{V_\varepsilon}(0)\right)}{\ln\,(1/\varsigma)} \leq M_2,$$

where $\mathrm{B}_1^{V_\varepsilon}(0)$ is the unit ball in V_ε of center 0.

(iii) *The family $\mathcal{E}_{\varepsilon,\delta}$ is Hölder continuous with respect to ε and δ, that is, there exist a positive constant M_3 and $\tau \in (0, \frac{1}{2}]$ such that*

$$\mathrm{dist}^{\mathrm{sym}}_{E_\varepsilon}(\mathcal{E}_{\varepsilon,\delta}, \mathcal{L}_{\varepsilon,\delta}\mathcal{E}_{0,0}) \leq M_3(\varepsilon+\delta)^\tau,$$

for all $0 < \varepsilon \leq 1$. In addition, there exist a positive constant M_4 and $\sigma \in (0, \frac{1}{2}]$ such that, for all $0 < \varepsilon \leq 1$, $0 < \delta \leq 1$,

$$\mathrm{dist}_{E_1}(\mathcal{E}_{\varepsilon,\delta}, \mathcal{L}_{\varepsilon,\delta}\mathcal{E}_{0,0}) \leq M_4(\varepsilon+\delta)^\sigma,$$

$$\text{and} \qquad \lim_{(\varepsilon,\delta)\to(0,0)} \mathrm{dist}_{E_1}(\mathcal{L}_{\varepsilon,\delta}\mathcal{E}_{0,0}, \mathcal{E}_{\varepsilon,\delta}) = 0.$$

Here ω, τ, σ and M_j are independent of ε and δ, and they can be computed explicitly.

Remark 1.4.1 *1.) Only Conditions* 2, 4, 5 *and B_ε not necessarily uniformly bounded (with respect to ε) in the E_1-norm, are needed in the construction of exponential attractors that satisfy* (i) *and* (ii) *of Theorem 1.2 (cf. [32, 58]).*

2.) To prove (iii) *of Theorem 1.2, again Conditions* 2 *and B_ε not necessarily uniformly bounded (with respect to ε) in the E_1-norm is required. In addition, it* *is sufficient to have E_1 replaced by E_ε in Condition 3 above (cf. [41]).*

Definition 1.8 (Inertial manifolds) *A set $\mathfrak{M}$ is called an inertial manifold for a semigroup $S(t)$ if:*

(i) *$\mathfrak{M}$ is a finite dimensional Lipschitz manifold in E;*

(ii) *$\mathfrak{M}$ is positively invariant under the flow, that is, $S(t)\mathfrak{M} \subset \mathfrak{M}, \ \forall t \geq 0$;*

(iii) *$\mathfrak{M}$ is exponentially attracting, that is, there exists a constant c_0 such that for every $u_0 \in E$, there exists a constant $c_1(u_0) > 0$ such that*

$$\mathrm{dist}_E(S(t)u_0, \mathfrak{M}) \leq c_1 e^{-c_0 t}, \ \ \forall t \geq 0.$$

Let us give a theorem that guarantees the existence of inertial manifolds for a semigroup $S(t)$ generated by an abstract evolution equation (cf. [68, 69, 71, 76]). Let F be a Banach space with $E \subset F$. Let us consider the abstract evolution equation

$$\frac{dU}{dt} + \mathcal{A}U = G(U), \quad U(0) = U_0, \tag{1.4.3}$$

where the nonlinear function $G : E \to F$ is assumed to be $\mathcal{C}^1$, globally bounded and Lipschitz continuous with Lipschitz constant k_G i.e.,

$$\|G(U)\|_F \leq M \quad \text{and} \quad \|G(U) - G(V)\|_F \leq k_G \|U - V\|_E, \quad \forall U, V \in E. \tag{1.4.4}$$

Assume the operator $\mathcal{A}$ admits a countable set of positive eigenvalues $\{\lambda_j\}_{j\geq 1}$, with a corresponding orthonormal system of eigenfunctions $\{\omega_j\}_{j\geq 1}$. The operator $\mathcal{A}$ is said to satisfy the *spectral gap* condition relative to G, if the point spectrum of $\mathcal{A}$ can be split into two parts σ_1 and σ_2, of which σ_1 is finite, and such that

$$\Lambda_1 := \sup\{\lambda \mid \lambda \in \sigma_1\}; \quad \Lambda_2 := \inf\{\lambda \mid \lambda \in \sigma_2\},$$

and $$E_i := \mathrm{Span}\{\omega_j \mid \lambda_j \in \sigma_i\},\ i = 1, 2,$$

then

$$\Lambda_1 - \Lambda_2 > k_G(\Lambda_1^\alpha + \Lambda_2^\alpha), \tag{1.4.5}$$

for some $0 \leq \alpha < 1$ independent of n, and the orthonormal decomposition $E = E_1 \oplus E_2$ holds, with continuous orthogonal projection $\mathcal{P} : E \to E_1$, and denote $\mathcal{Q} = I - \mathcal{P} : E \to E_2$. Observe that E_1 is finite dimensional.

Assume that the evolution equation (1.4.3) generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, $S(t) : E \to E$, with $S(t)F \subset E, \quad \forall t \geq 0$. Assume that the projectors above defines an exponential dichotomy of $\{e^{-\mathcal{A}t}\}_{t\geq 0}$, i.e., there exist k_1, k_2 independent of n

$$\begin{cases} \|e^{-\mathcal{A}t}\mathcal{P}\|_{\mathcal{L}(E)} \leq k_1 e^{-\Lambda_1 t}, & \forall t \leq 0, \\ \|e^{-\mathcal{A}t}\mathcal{P}\|_{\mathcal{L}(F,E)} \leq k_1 \Lambda_1^\alpha e^{-\Lambda_1 t}, & \forall t \leq 0, \\ \|e^{-\mathcal{A}t}\mathcal{Q}\|_{\mathcal{L}(E)} \leq k_2 e^{-\Lambda_2 t}, & \forall t \geq 0, \\ \|e^{-\mathcal{A}t}\mathcal{Q}\|_{\mathcal{L}(F,E)} \leq k_2 \left(\dfrac{1}{t^\alpha} + \Lambda_2^\alpha\right) e^{-\Lambda_2 t}, & \forall t > 0. \end{cases} \tag{1.4.6}$$

The following theorem, proven in [68] (see also [71]), states that the spectral gap condition is a sufficient condition for the existence of an inertial manifold.

Theorem 1.3 ***(cf. [76])*** *Let $\mathcal{A}$ be a densely defined operator generating a continuous semigroup on a separable Hilbert space E. Let G be $\mathcal{C}^1$ and satisfy condition* (1.4.4)*. Assume $\mathcal{A}$ satisfies the spectral gap condition* (1.4.5) *and* (1.4.6)*. Then the semigroup $S(t)$ generated by the evolution equation* (1.4.3) *admits an* *inertial manifold $\mathfrak{M}$ in E, of the form*

$$\mathfrak{M} = \text{graph}(\Phi) := \{p + \Phi(p) \mid p \in E_1\}, \tag{1.4.7}$$

and $\Phi : E_1 \to E_2$ is of class $\mathcal{C}^1$.

Lemma 1.1 ***(cf. [68, Thm. 8.1])*** *Let X, Y and Z be Banach spaces such that the embedding $X \hookrightarrow Y$ is compact ($X \subset\subset Y$) and $Y \subset Z$ is continuous. Assume that X is reflexive. Assume that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$, and $\dfrac{du_n}{dt}$ is uniformly bounded in $L^p(0,T;Z)$, for some $p > 1$. Then there exists a subsequence that converges strongly in $L^2(0,T;Y)$ and a.e in $\Omega \times (0,T)$.*

Lemma 1.2 ***(cf. [71, Lemma 3.3, Chap. III])*** *Let Y and Z be Banach spaces such that the injection $Y \hookrightarrow Z$ is continuous. If $\psi \in L^\infty(0,T;Y)$ and is weakly continuous with values in Z, then ψ is weakly continuous with values in Y.*

Lemma 1.3 (Generalized Hölder's inequality) *If $p_1, p_2, \cdots, p_n$ are such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$$

and $f_j \in L^{p_j}(\Omega)$, then their product $f_1 f_2 \cdots f_n \in L^1(\Omega)$ and

$$\int_\Omega |f_1(x) f_2(x) \cdots f_n(x)| dx \leq \|f_1\|_{L^{p_1}(\Omega)} \cdots \|f_n\|_{L^{p_n}(\Omega)}.$$

Lemma 1.4 (Young's inequality) *Let a and b be positive real numbers. For any $\epsilon > 0$, there holds*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad p \geq 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad C_\varepsilon = \frac{(\varepsilon p)^{-q/p}}{q}.$$

Lemma 1.5 (Gronwall and Uniform Gronwall's Lemmas) *Let* g, h, f *be three positive and locally integrable functions over* $(t_0, +\infty)$ *such that* f' *locally integrable over* $(t_0, +\infty)$, *and satisfy:*

$$\frac{df}{dt} \leq gf + h, \quad \forall t \geq t_0.$$

Then

$$f(t) \leq f(t_0) \exp\left(\int_{t_0}^{t} g(\sigma) d\sigma\right) + \int_{t_0}^{t} h(s) \exp\left(-\int_{t}^{s} g(\sigma) d\sigma\right) ds, \quad \forall t \geq t_0.$$

If in addition,

$$\int_{t}^{t+r} g(s) ds \leq \gamma_1, \quad \int_{t}^{t+r} h(s) ds \leq \gamma_2, \quad \int_{t}^{t+r} f(s) ds \leq \gamma_3, \quad t \geq t_0,$$

where $r, \gamma_1, \gamma_2, \gamma_3$ *are positive constants, then*

$$f(t+r) \leq \left(\frac{\gamma_3}{r} + \gamma_2\right) e^{\gamma_1}, \quad \forall t \geq t_0.$$

Lemma 1.6 (Generalized Gronwall's Lemma [45]) *Let* Θ *be an absolutely continuous positive function on* $[0, \infty)$, *which satisfies for some* $\nu > 0$ *the differential inequality*

$$\frac{d}{dt}\Theta(t) + 2\nu\Theta(t) \leq f(t)\Theta(t) + h(t)$$

for almost every $t \in [0, \infty)$, *where* f *and* h *are functions on* $[0, \infty)$ *such that*

$$\int_{\tau}^{t} |f(s)| ds \leq \alpha(1 + (t - \tau)^{\omega}), \qquad \sup_{t \geq 0} \int_{t}^{t+1} |h(s)| ds \leq \beta,$$

for some $\alpha, \beta \geq 0$ *and* $\omega \in [0, 1)$. *Then*

$$\Theta(t) \leq \Lambda\Theta(0)e^{-\nu t} + K, \quad \forall t \geq 0,$$

for some $\Lambda = \Lambda(\alpha, \omega) \geq 1$ *and* $K = K(\alpha, \omega, \beta) \geq 0$.

Lemma 1.7 (Agmon's inequality) *Let* $\Omega \subset \mathbb{R}^n$ *be an open set of class* $\mathcal{C}^n$. *There exists a constant* $c > 0$ *which depends on* Ω *such that:*

$$\|u\|_{L^\infty(\Omega)} \leq \begin{cases} c\|u\|^{1/2}_{H^{(n-2)/2}(\Omega)}\|u\|^{1/2}_{H^{(n+2)/2}(\Omega)}, & \forall u \in H^{(n+2)/2}(\Omega) \text{ if } n \text{ is even,} \\ c\|u\|^{1/2}_{H^{(n-1)/2}(\Omega)}\|u\|^{1/2}_{H^{(n+1)/2}(\Omega)}, & \forall u \in H^{(n+1)/2}(\Omega) \text{ if } n \text{ is odd.} \end{cases}$$

Lemma 1.8 (Interpolation inequalities) *Let* $s_1 \leq s \leq s_2$ *such that* $s = \theta s_1 + (1-\theta)s_2, \theta \in [0,1]$. *There exists* $c > 0$ *such that*

$$\|u\|_{H^s(\Omega)} \leq \|u\|^{1-\theta}_{H^{s_1}(\Omega)}\|u\|^{\theta}_{H^{s_2}(\Omega)}, \quad \forall u \in H^{s_2}(\Omega).$$

More generally, let $p \in [1, \infty)$, $m \geq 1$ *and* $\varphi \in W^{m,p}(\Omega)$. *Then, there exists a constant* $C > 0$ *and* $\mu = \frac{n}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$ *such that*

$$\|u\|_{L^r(\Omega)} \leq C\|u\|^{1-\mu}_{L^p(\Omega)}\|u\|^{\mu}_{W^{m,p}(\Omega)}, \quad r \in \begin{cases} [p, \infty] & \text{if } m - \dfrac{n}{p} > 0, \\ [p, \infty) & \text{if } m - \dfrac{n}{p} = 0, \\ [p, -\dfrac{n}{m-(n/p)}] & \text{if } m - \dfrac{n}{p} < 0. \end{cases}$$

1.5 Functional settings

Denote

$$m(\varphi) = \frac{1}{|\Omega|}\int_\Omega \varphi(x)\,dx \quad \text{and} \quad \bar{\varphi} = \varphi - m(\varphi).$$

If W is a Sobolev-type space, then we set

$$\dot{W} = \{\varphi \in W, \, m(\varphi) = 0\}.$$

We denote by W' the dual space of W.

Throughout Chapters 2, 3 and 4, all the problems considered are subject to the boundary conditions either of Neumann or periodic type

$$\partial_n \phi|_{\partial\Omega} = \partial_n \Delta\phi|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0, \tag{1.5.1}$$

(the symbol ∂_n denotes the outward normal derivative) if Ω is a bounded domain of $\mathbb{R}^d$, with smooth boundary $\partial\Omega$, or

$$\begin{cases} u|_{x_i=0} = u|_{x_i=L_i},\ u_{x_i}|_{x_i=0} = u_{x_i}|_{x_i=L_i},\ i = 1, ..., d, \\ \phi|_{x_i=0} = \phi|_{x_i=L_i},\ i = 1, ..., d, \\ \text{for } \phi \text{ and the derivatives of } \phi \text{ of order} \leq 3, \end{cases} \tag{1.5.2}$$

if $\Omega = \Pi_{i=1}^d (0, L_i)$.

Let us define the linear unbounded operator, with domain $\mathscr{D}(N)$,

$$N = -\Delta : \mathscr{D}(N) \to \dot{L}^2(\Omega),$$

with

$$\mathscr{D}(N) = \begin{cases} \left\{\varphi \in H^2(\Omega),\, \partial_n \varphi|_{\partial\Omega} = 0\right\}, & \text{in case of (1.5.1)}, \\ H^2_{per}(\Omega), & \text{in case of (1.5.2)}, \end{cases}$$

which is self-adjoint and nonnegative. If N is restricted to $\mathscr{D}(N) \cap \dot{L}^2(\Omega)$, then it turns to be positive with compact inverse N^{-1}. Moreover, one can define the powers N^r of N for $r \in \mathbb{R}$ (cf. [71]). The spaces $V_r = \mathscr{D}(N^{r/2})$ are Hilbert spaces. In particular, $V_{-1} = (H^1(\Omega))'$ or $(H^1_{per}(\Omega))'$, $V_0 = L^2(\Omega)$, $V_1 = H^1(\Omega)$ or $H^1_{per}(\Omega)$. The injection $V_{r_1} \hookrightarrow V_{r_2}$ is compact whenever $r_1 > r_2$. We denote by $\|.\|$ and $(.,.)$ the usual norm and scalar product in $L^2(\Omega)$ (and also in $L^2(\Omega)^d$). When r is positive, V_r is a subspace of $H^r(\Omega)$ and

$$\|\varphi\|_r = \left(\|N^{r/2}\varphi\|^2 + |m(\varphi)|^2\right)^{1/2}$$

is a norm on V_r which is equivalent to the usual $H^r(\Omega)-$norm; we endow V_r' with the norm

$$\|\varphi\|_{-r} = \left(\|N^{-r/2}\bar{\varphi}\|^2 + |m(\varphi)|^2\right)^{1/2}.$$

Also, throughout this thesis, we denote the function $G(s) = \int_0^s g(\varsigma)d\varsigma$ and we assume that $g \in \mathcal{C}^2(\mathbb{R})$ and the following conditions hold (cf., e.g., [7]):

$$G(s) \geq -C_1, \;\; C_1 \geq 0, \;\; \forall s \in \mathbb{R}, \tag{1.5.3}$$

$$\forall \gamma \in \mathbb{R}, \;\exists\; C_2(\gamma) > 0, C_3(\gamma) \geq 0 \text{ such that}$$

$$(s-\gamma)g(s) - C_2 G(s) \geq -C_3, \;\; \forall s \in \mathbb{R}, \tag{1.5.4}$$

$$\text{(where } C_2, C_3 \text{ are bounded when } \gamma \text{ is bounded)}$$

$$g'(s) \geq -C_4, \;\; C_4 \geq 0, \;\; \forall s \in \mathbb{R}, \tag{1.5.5}$$

$$|g''(s)| \leq C_5\left(|s|^p + 1\right), \;\; C_5 > 0, \;\; \forall s \in \mathbb{R}. \tag{1.5.6}$$

We note that in space dimension one no growth assumption on g is needed.

The space $\overline{X}^Y$ denotes the closure of a metric space $X \subset Y$ in the topology of the complete metric space Y. Furthermore, there exist two positive constants C_6, C_7 such that

$$\|\varphi\|_{-1} \leq C_6\|\varphi\| \;\; \forall \varphi \in L^2(\Omega),$$

$$\|\bar{\varphi}\| \leq C_7\|\nabla\varphi\|, \;\; \forall \varphi \in V_1.$$

For every $r \geq 0$, we endow the Hilbert spaces

$$\mathcal{U}_r = \mathscr{D}(N^{r/2}) \times \mathscr{D}(N^{(r-1)/2}) \quad \text{and} \quad \mathcal{W}_r = \mathscr{D}(N^{r/2}) \times \mathscr{D}(N^{r/2})$$

respectively with the norms

$$\|(\varphi,\psi)\|_{\mathcal{U}_r} = \left(\|(I+N)^{r/2}\varphi\|^2 + \|(I+N)^{(r-1)/2}\psi\|^2\right)^{1/2}$$

$$\text{and} \quad \|(\varphi,\psi)\|_{\mathcal{W}_r} = \left(\|(I+N)^{r/2}\varphi\|^2 + \|(I+N)^{r/2}\psi\|^2\right)^{1/2}.$$

Note that $\|(I+N)^{r/2}.\|^2$ is a norm on V_r which is equivalent to $\|.\|_r$. Sometimes, we will use the equivalent norms

$$\|(\varphi,\psi)\|_{\mathcal{U}_{r,\varepsilon}} = \left(\|(I+N)^{r/2}\varphi\|^2 + \varepsilon\|(I+N)^{(r-1)/2}\psi\|^2\right)^{1/2}$$

$$\text{and} \quad \|(\varphi,\psi)\|_{\mathcal{W}_{r,\epsilon}} = \left(\|(I+N)^{r/2}\varphi\|^2 + \epsilon\|(I+N)^{r/2}\psi\|^2 + \|(I+N)^{(r-1)/2}\psi\|^2\right)^{1/2}.$$

Then, we set

$$K_\alpha = \{\varphi \in L^2(\Omega),\ |m(\varphi)| \le \alpha\},$$

$$K_{\alpha,\sigma} = \{(\varphi,\psi) \in \mathcal{U}_1,\ |m(\varphi)| \le \alpha,\ |m(\psi)| \le \sigma\},$$

$$\widetilde{K}_{\alpha,\rho} = \{(\varphi,\psi) \in \mathcal{W}_1,\ |m(\varphi)| \le \alpha,\ |m(\psi)| \le \rho\},$$

for some $\alpha, \rho, \sigma \ge 0$.

The following inequalities hold, there exists $c > 0$ such that for any $\alpha \in [0,1]$:

$$\|(\tau I + \delta N)^{-\alpha} N^\alpha q\| \le \frac{1}{\delta^\alpha}\|q\|, \quad \forall q \in \mathcal{D}(N^\alpha), \tag{1.5.7}$$

$$\|(\tau I + \delta N)^{-\alpha} w\| \le \frac{1}{\tau^\alpha}\|w\|, \quad \forall q \in L^2(\Omega), \tag{1.5.8}$$

$$\|\nabla(\tau I + \delta N)^{-1/2} u\| \ge c\|\bar{u}\|, \qquad \forall u \in L^2(\Omega). \tag{1.5.9}$$

We multiply $(1.2.5)_1$ and $(1.2.6)_2$ by 1 and integrate over Ω, then we find

$$\frac{d}{dt}m(\phi) = 0 \qquad \text{and} \qquad \frac{d}{dt}[m(\phi) + \varepsilon m(u)] = 0,$$

respectively, so that

$$m(\phi(t)) = m(\phi_0) \qquad \text{and} \qquad m(u(t)) = m(u_0), \quad \forall t \ge 0. \tag{1.5.10}$$

Performing same computation on (1.2.12) and (1.2.13), we also arrive at (1.5.10).

We remind also the following properties of the operator N. The eigenvalues

$\{\lambda_k\}$ and corresponding eigenvectors $\{e_k\}$ (which are orthogonal basis of $L^2(\Omega)$) of the operator $N : \{\varphi \in H^2(\Omega),\ \partial_n\varphi|_{\partial\Omega} = 0\} \to \dot{L}^2(\Omega)$ is given respectively by the form:

$$\lambda = \pi^2 \frac{k_1^2}{L_1^2}, \text{ and } e(x) = \sqrt{\frac{2}{L_1}} \cos\frac{\pi k_1 x_1}{L_1} \quad \text{if} \quad d = 1, \tag{1.5.11}$$

and for $N : H^2_{per}(\Omega) \to \dot{L}^2(\Omega)$

$$\lambda = \begin{cases} 4\pi^2 \dfrac{k_1^2}{L_1^2}, & \text{if} \quad d = 1, \\ 4\pi^2 \left(\dfrac{k_1^2}{L_1^2} + \dfrac{k_2^2}{L_2^2}\right), & \text{if} \quad d = 2, \end{cases} \tag{1.5.12}$$

and

$$e(x) = \begin{cases} \dfrac{1}{\sqrt{L_1}},\ \sqrt{\dfrac{2}{L_1}} \cos\dfrac{2\pi k_1 x_1}{L_1}, \quad \sqrt{\dfrac{2}{L_1}} \sin\dfrac{2\pi k_1 x_1}{L_1}, & \text{if} \ \ d = 1, \\ \dfrac{1}{\sqrt{L_1 L_2}},\ \sqrt{\dfrac{2}{L_1 L_2}} \cos\dfrac{\pi k x}{L}, \quad \sqrt{\dfrac{2}{L_1 L_2}} \sin\dfrac{2\pi k x}{L}, & \text{if} \quad d = 2, \end{cases} \tag{1.5.13}$$

with $\dfrac{kx}{L} = \dfrac{k_1 x_1}{L_1} + \dfrac{k_2 x_2}{L_2}$, for $k_1, k_2 = 0, 1, 2, ...$

For any $\eta > 0$, we denote by $\mathcal{C}_\eta((-\infty, 0]; \mathcal{U}_d)$ the following Banach space (cf., e.g., [22, 62])

$$\mathcal{C}_\eta((-\infty, 0]; \mathcal{U}_d) = \{f,\ f : (-\infty, 0] \to \mathcal{U}_d \text{ continuous and } \sup_{t\le 0} e^{\eta t}\|f(t)\|_{\mathcal{U}_d} < \infty\}, \tag{1.5.14}$$

with norm

$$\|f\|_{\mathcal{C}_\eta((-\infty,0];\mathcal{U}_d)} = \sup_{t\leq 0} e^{\eta t}\|f(t)\|_{\mathcal{U}_d}.$$

Throughout this work, the same letter c, C and c' (and sometimes c_i, $i = 0, 1, 2, ...$) denote positive constants that may change from line to line, but are always independent of ϵ, ε, δ and on time (unless explicitly specified).

1.6 Publications

The following articles have been published from this thesis.

[1] A. Bonfoh and C.D. Enyi, Large time behavior of a conserved phase-field system, *Communications on Pure and Applied Analysis*, **15** (2016), 1077-1105.

[2] A. Bonfoh and C.D. Enyi, The Cahn-Hilliard equation as limit of a conserved phase-field system, *Asymptotic Analysis*, **101** (2017), 97-148.

CHAPTER 2

VISCOUS CAHN-HILLIARD EQUATION AS A LIMIT OF A CONSERVED PHASE-FIELD SYSTEM

We consider the following problem

$$\begin{cases} \tau\phi_t + N(\delta\phi_t + N\phi + g(\phi) - u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \varepsilon u_t + \phi_t + Nu = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \tag{2.0.1}$$

where $\delta > 0$ is fixed, $\varepsilon \in (0, 1]$, g satisfies (1.5.3)-(1.5.6), with $p > 0$ arbitrary when $d = 1, 2$ and $p \in [0, 3]$ when $d = 3$.

2.1 A priori estimates

We multiply $(2.0.1)_1$ by $N^{-1}\phi_t$ and $(2.0.1)_2$ by u and we integrate over Ω, then summing the resulting equations, we obtain

$$\frac{d}{dt}\left(\varepsilon\|u\|^2 + \|\nabla\phi\|^2 + 2\int_\Omega G(\phi)dx\right) + 2\|\nabla u\|^2 + 2\tau\|\phi_t\|_{-1}^2 + 2\delta\|\phi_t\|^2 = 0. \tag{2.1.1}$$

Next, we multiply $(2.0.1)_1$ by $N^{-1}\bar{\phi}$ and we integrate over Ω, and we obtain,

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|\bar{\phi}\|_{-1}^2 + \delta\|\bar{\phi}\|^2\right) + \|\nabla\phi\|^2 + \int_\Omega g(\phi)\bar{\phi}dx = (u, \bar{\phi}). \tag{2.1.2}$$

Owing to (1.5.4), taking $\gamma = m(\phi_0)$, we have

$$\int_\Omega g(\phi)\bar{\phi}dx \geq C_2\int_\Omega G(\phi)dx - |\Omega|C_3.$$

Now, applying Young's inequality to $(u, \bar{\phi})$ and noting that $\|\bar{\phi}\| \leq c\|\nabla\phi\|$, (2.1.2) gives

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|\bar{\phi}\|_{-1}^2 + \delta\|\bar{\phi}\|^2\right) + \frac{1}{2}\|\nabla\phi\|^2 + C_2\int_\Omega G(\phi)dx \leq |\Omega|C_3 + c\|u\|^2, \tag{2.1.3}$$

then we obtain

$$\frac{d}{dt}\left(\tau\|\bar{\phi}\|_{-1}^2 + \delta\|\bar{\phi}\|^2\right) + \|\phi\|_1^2 + c_1\int_\Omega G(\phi)dx \leq c\|u\|_1^2 + c_2, \tag{2.1.4}$$

where $c_2 = c(m(\phi_0)) = \frac{1}{2}|m(\phi_0)|^2 + |\Omega|C_3$, but we will omit the dependence of constants with respect to $m(\phi_0)$.

We multiply $(2.0.1)_1$ by ϕ and we integrate over Ω, and we obtain

$$\frac{d}{dt}\left(\tau\|\phi\|^2 + \delta\|\nabla\phi\|^2\right) + \|N\phi\|^2 + (\nabla g(\phi), \nabla\phi) - (\nabla u, \nabla\phi) = 0. \tag{2.1.5}$$

From (1.5.5), we have

$$
\begin{aligned}
&(\nabla g(\phi), \nabla\phi) = \int_\Omega g'(\phi)|\nabla\phi|^2 \geq -C_4\|\nabla\phi\|^2,\\
&|(\nabla u, \nabla\phi)| \leq \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla\phi\|^2,
\end{aligned}
$$

so that (2.1.5) gives

$$
\frac{d}{dt}\left(\tau\|\phi\|^2 + \delta\|\nabla\phi\|^2\right) + \|N\phi\|^2 \leq c\left(\|u\|_1^2 + \|\phi\|_1^2\right). \tag{2.1.6}
$$

Summing (2.1.1), ϖ_1(2.1.4) and ϖ_2(2.1.6) with $\varpi_1, \varpi_2 > 0$, and adding $2|m(u_0)|^2$ on both sides of the resulting inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}E(t) + 2\tau\|\phi_t\|_{-1}^2 + 2\delta\|\phi_t\|^2 + 2\|u\|_1^2 + \varpi_1\|\phi\|_1^2 + \varpi_2\|N\phi\|^2 + \varpi_1 c_1\int_\Omega G(\phi)dx\\
&\leq 2|m(u_0)|^2 + \varpi_1 c_2 + c(\varpi_1 + \varpi_2)\|u\|_1^2 + c\varpi_2\|\phi\|_1^2,
\end{aligned}
\tag{2.1.7}
$$

where

$$
\begin{aligned}
E(t) =&\varepsilon\|u\|^2 + \|\nabla\phi\|^2 + 2\int_\Omega G(\phi)dx\\
&+ \varpi_1\left(\tau\|\bar{\phi}\|_{-1}^2 + \delta\|\bar{\phi}\|^2\right) + \varpi_2\left(\tau\|\phi\|^2 + \delta\|\nabla\phi\|^2\right).
\end{aligned}
$$

Now, we choose ϖ_1 and ϖ_2 such that $2 - c(\varpi_1 + \varpi_2) > 0$ and $\varpi_1 - c\varpi_2 > 0$, indeed, we can choose $\varpi_1 = \dfrac{2}{2c+3}$ and $\varpi_2 = \dfrac{1}{(c+1)(2c+3)}$. We deduce that

$$
\frac{d}{dt}E(t) + c\left(\|u\|_1^2 + \|\phi\|_2^2 + \int_\Omega G(\phi)dx + \|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2\right) \leq c', \tag{2.1.8}
$$

where $c' = c(m(\phi_0), m(u_0)) = \varpi_1 c_2 + 2|m(u_0)|^2 + \varpi_2|m(\phi_0)|^2$.

Due to (1.5.3), there exists $c_1 > 0$ (independent of ε) such that

$$
\begin{aligned}
E(t) &\geq \varepsilon\|u\|^2 + \varpi_2\left(\tau\|\phi\|^2 + \delta\|\nabla\phi\|^2\right) - 2C_1|\Omega|\\
&\geq c_1(\|\phi\|_1^2 + \varepsilon\|u\|^2) - 2C_1|\Omega|.
\end{aligned}
\tag{2.1.9}
$$

Form (1.5.4) and (1.5.5), choosing $\gamma = 0$, we have $G(s) \leq c(sg(s) + 1)$,

Furthermore, from (1.5.6) and the Mean Value Theorem, we deduce that

$$|g'(s)| \leq c(1 + |s|^{p+1}) \quad \text{and} \quad |g(s)| \leq c(1 + |s|^{p+2}).$$

Therefore, using Young's inequality and exploiting the Sobolev embedding of $H^1(\Omega)$ into $L^{p+3}(\Omega)$, then due to (1.5.4) and (1.5.6) we obtain that

$$\int_\Omega G(\phi)dx \leq c(1 + \|\phi\|^{p+3}_{L^{p+3}(\Omega)}) \leq c(1 + \|\phi\|_1^{p+3}). \tag{2.1.10}$$

Hence

$$E(t) \leq c(\|\phi\|_1^{p+3} + \varepsilon\|u\|^2 + 1). \tag{2.1.11}$$

It follows from (2.1.9) and (2.1.11) that there exist $c_0', c_1, c_2 \geq 0$, independent of ε, such that

$$c_1\|(\phi(t), u(t))\|^2_{\mathcal{U}_{1,\varepsilon}} - c_0' \leq E(t) \leq c_2\left(\varepsilon\|u(t)\|^2 + \|\phi(t)\|_1^{p+3} + 1\right), \tag{2.1.12}$$

Now, we deduce from (2.1.8) that

$$\begin{aligned} &\frac{d}{dt}E(t) + \frac{c}{2}\left(\|u\|_1^2 + \|\phi\|_2^2 + \int_\Omega G(\phi)dx + \|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2\right) \\ &+ \frac{c}{2}\left(\|u\|^2 + \|\phi\|_1^2 + \int_\Omega G(\phi)dx\right) \leq c'. \end{aligned} \tag{2.1.13}$$

Observe that

$$E(t) \leq c\left(\|\phi\|_1^2 + \varepsilon\|u\|^2 + \int_\Omega G(\phi)dx\right),$$

so that there exists $c_0 > 0$ (independent of ε) such that

$$\|\phi\|_1^2 + \|u\|^2 + \int_\Omega G(\phi)dx \geq c_0 E(t). \tag{2.1.14}$$

Finally, we obtain from (2.1.13) and (2.1.14) that

$$\frac{d}{dt}E(t) + c_0 E(t) + c\left(\|u\|_1^2 + \|\phi\|_2^2 + \|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2\right) \le c. \tag{2.1.15}$$

Now, we multiply $(2.0.1)_1$ by ϕ_t and $N\phi$, then $(2.0.1)_2$ by Nu and we integrate over Ω, and we obtain respectively,

$$\frac{1}{2}\frac{d}{dt}\|N\phi\|^2 + \tau\|\phi_t\|^2 + \delta\|\nabla\phi_t\|^2 + (Ng(\phi), \phi_t) - (\nabla u, \nabla\phi_t) = 0, \tag{2.1.16}$$

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|\nabla\phi\|^2 + \delta\|N\phi\|^2\right) + \|\nabla N\phi\|^2 + (g'(\phi)\nabla\phi, \nabla N\phi) - (\nabla u, \nabla N\phi) = 0, \tag{2.1.17}$$

and $$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2 + \|Nu\|^2 + (\nabla\phi_t, \nabla u) = 0. \tag{2.1.18}$$

Summing (2.1.16), (2.1.17) and (2.1.18), we get

$$\frac{d}{dt}\left(\varepsilon\|\nabla u\|^2 + \|N\phi\|^2 + \tau\|\nabla\phi\|^2 + \delta\|N\phi\|^2\right) + \|\nabla N\phi\|^2 + \|Nu\|^2 + 2\tau\|\phi_t\|^2$$
$$+ \delta\|\nabla\phi_t\|^2 \le |(\nabla g(\phi), \nabla N\phi)| + |(Ng(\phi), \phi_t)| + |(Nu, N\phi)|. \tag{2.1.19}$$

We have $(Ng(\phi), \phi_t) = (g'(\phi)\nabla\phi, \nabla\phi_t)$.

When $d = 1$, we have

$$\begin{aligned}|(g'(\phi)\nabla\phi, \nabla\phi_t)| &\le \|g'(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|\|\nabla\phi_t\| \\ &\le (2\delta)^{-1}\|g'(\phi)\|_{L^\infty(\Omega)}^2\|\phi\|_1^2 + \frac{\delta}{2}\|\nabla\phi_t\|^2. \end{aligned}\tag{2.1.20}$$

When $d = 2$, we have by (1.5.6) and Young's inequality

$$\begin{aligned}|(g'(\phi)\nabla\phi, \nabla\phi_t)| &\le \int_\Omega (1 + |\phi|^{p+1})|\nabla\phi||\nabla\phi_t| dx \\ &\le c\left(1 + \|\phi\|_{L^{4(p+1)}(\Omega)}^{p+1}\right)\|\nabla\phi\|_{L^4(\Omega)^2}\|\nabla\phi_t\| \\ &\le c\left(1 + \|\phi\|_{L^{4p+4}(\Omega)}^{p+1}\right)\|\nabla\phi\|_1\|\nabla\phi_t\| \\ &\le c\delta^{-1}(1 + \|\phi\|_1^{2p+2})\|\phi\|_2^2 + \frac{\delta}{2}\|\nabla\phi_t\|^2. \end{aligned}\tag{2.1.21}$$

When $d = 3$, using (1.5.6), Agmon's inequality and Young's inequality, we have

$$
\begin{aligned}
|(g'(\phi)\nabla\phi, \nabla\phi_t)| &\le \int_\Omega (1 + |\phi|^4)|\nabla\phi||\nabla\phi_t|dx \\
&\le \|\nabla\phi\|\|\nabla\phi_t\| + \int_\Omega |\phi|^4|\nabla\phi||\nabla\phi_t| \\
&\le \|\nabla\phi\|\|\nabla\phi_t\| + c\|\phi\|^2_{L^\infty(\Omega)}\|\phi\|^2_{L^6(\Omega)}\|\nabla\phi\|_{L^6(\Omega)^2}\|\nabla\phi_t\| \\
&\le c\|\nabla\phi\|\|\nabla\phi_t\| + c\|\phi\|_1^3\|\phi\|_2^2\|\nabla\phi_t\| \\
&\le c\delta^{-1}\left(1 + \|\phi\|_1^6\right)(1 + \|\phi\|_2^2)\|\phi\|_2^2 + \frac{\delta}{2}\|\nabla\phi_t\|^2. \qquad (2.1.22)
\end{aligned}
$$

The term $|(g'(\phi)\nabla\phi, \nabla N\phi)|$ is treated in the same way. Also, Young's inequality gives that

$$
\begin{aligned}
|(Nu, N\phi)| &\le \frac{1}{2}\|Nu\|^2 + \frac{1}{2}\|N\phi\|^2 \\
&\le \frac{1}{2}\|Nu\|^2 + c(1 + \|\phi\|_2^2)\|\phi\|_2^2.
\end{aligned}
$$

Hence, adding $|m(\phi_0)|^2 + |m(u_0)|^2$ to both sides of (2.1.19), we deduce

$$
\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|u\|_1^2 + \|\phi\|_2^2 + \tau\|\phi\|_1^2 + \delta\|\phi\|_2^2\right) + \|\phi\|_3^2 + \|u\|_2^2 + 2\tau\|\phi_t\|^2 + \delta\|\phi_t\|_1^2 \\
&\le \frac{M_1(t)}{\delta}\left(\|\phi\|_2^2 + 1\right) + c, \qquad (2.1.23)
\end{aligned}
$$

where $c = |m(\phi_0)|^2 + |m(u_0)|^2$ and

$$
M_1(t) = \begin{cases} c\|g'(\phi)\|^2_{L^\infty(\Omega)}, & \text{if } d = 1, \\ c\left(\|\phi\|_1^{2p+2} + 1\right), & \text{if } d = 2, \\ c\left(\|\phi\|_1^6 + 1\right), & \text{if } d = 3. \end{cases} \qquad (2.1.24)
$$

We note that the function $t \mapsto M_1(t)\left(\|\phi\|_2^2 + 1\right)$ is $L^1(0,T)$ since $t \mapsto M_1(t)$ is $\mathcal{C}([0,T])$ and $\phi \in L^2(0,T;V_2)$ holds from (2.1.15).

2.2 Well-posedness

We start the proof by giving the weak formulation of problem (2.0.1). Let $(\phi_0, u_0) \in \mathcal{U}_1$.

Weak formulation: For any $T > 0$, find $(\phi, u) : [0, T] \mapsto \mathcal{U}_2$ such that

$$\phi(0) = \phi_0, \quad u(0) = u_0, \tag{2.2.1}$$

and for almost every $t \in [0, T]$ and all $(q, v) \in \mathcal{U}_2$,

$$\begin{aligned} &\frac{d}{dt}\big[\tau(\phi, q) + \delta(\nabla\phi, \nabla q)\big] + (\Delta\phi, \Delta q) + (\nabla g(\phi), \nabla q) - (\nabla u, \nabla q) = 0, \\ &\frac{d}{dt}\big[\varepsilon(u, v) + (\phi, v)\big] + (\nabla u, \nabla v) = 0. \end{aligned} \tag{2.2.2}$$

Theorem 2.1 *We assume that* (1.5.3)-(1.5.6) *hold. If* $(\phi_0, u_0) \in \mathcal{U}_1$*, then* (2.0.1) *possesses a unique solution* (ϕ, u) *such that*

$$(\phi, u) \in \mathcal{C}([0, T]; \mathcal{U}_1) \cap L^2(0, T; \mathcal{U}_2), \quad m(\phi) = m(\phi_0), \;\; m(u) = m(u_0),$$

for any $T > 0$*. Moreover, if* $(\phi_0, u_0) \in \mathcal{U}_2$*, then*

$$(\phi, u) \in \mathcal{C}([0, T]; \mathcal{U}_2) \cap L^2(0, T; \mathcal{U}_3).$$

Proof. The proof is by the Faedo-Galerkin method.

(i) There is a complete orthonormal family e_j on $L^2(\Omega)$ made of eigenvectors of $-\Delta$ such that $-\Delta e_j = \lambda_j e_j, \quad e_j \in \mathscr{D}(N),\ j = 0, 1, 2, 3, \cdots$, (in fact, we have that $e_j \in \mathscr{W}$, since Ω is sufficiently regular) and $0 = \lambda_0, \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$, where $\mathscr{W} := \{\varphi \in H^4(\Omega),\ \partial_n \varphi|_{\partial\Omega} = \partial_n \Delta\varphi|_{\partial\Omega} = 0\}$ or $\mathscr{W} = H^4_{per}(\Omega)$. We set $E_m = span\{e_0, e_1, e_2, \cdots, e_m\}$ and P_m is the orthonormal projection on E_m. We consider the approximate problem:

For each m, find an approximate solution (ϕ_m, u_m) of the form $\phi_m(t) = \sum_{j=0}^{m} \alpha_j(t) e_j$ and $u_m(t) = \sum_{j=0}^{m} \beta_j(t) e_j$ satisfying

$$
\begin{aligned}
&\frac{d}{dt}\big[\tau(\phi_m, e_j) + \delta(\nabla\phi_m, \nabla e_j)\big] + (\Delta\phi_m, \Delta e_j) + (g'(\phi_m)\nabla\phi_m, \nabla e_j) - (\nabla u_m, \nabla e_j) = 0,\\
&\frac{d}{dt}\big[\varepsilon(u_m, e_j) + (\phi_m, e_j)\big] + (\nabla u_m, \nabla e_j) = 0, \qquad j = 0, 1, \cdots, m,\\
&\phi_m(0) = P_m\phi_0, \quad u_m(0) = P_m u_0,
\end{aligned}
\tag{2.2.3}
$$

where

$$
P_m\phi_0 = \phi_{0m} \to \phi_0 \quad \text{and} \quad P_m u_0 = u_{0m} \to u_0. \tag{2.2.4}
$$

Problem (2.2.3) yields

$$
\begin{aligned}
&(\tau + \delta\lambda_j)\alpha_j' + \lambda_j^2\alpha_j + (g'(\phi_m)\nabla\phi_m, \nabla e_j) - \lambda_j\beta_j = 0\\
&\varepsilon\beta_j' + \alpha_j' + \lambda_j\beta_j = 0, \qquad j = 0, 1, \cdots, m.
\end{aligned}
\tag{2.2.5}
$$

We set $Y = (\alpha_0, \alpha_1, \cdots, \alpha_m)$ and $Z = (\beta_0, \beta_1, \cdots, \beta_m)$, thus we deduce from (2.2.5) the system:

$$
\begin{aligned}
&(\tau I + \delta M)Y' + M^2 Y + G(Y) - MZ = 0,\\
&\epsilon(\tau I + \delta M)Z' + \big((1+\tau)I + \delta M\big)Z - M^2 Y - G(Y) = 0,
\end{aligned}
\tag{2.2.6}
$$

where

$$
I = (\delta_{ij})_{i,j=0,1,\cdots,m}, \qquad M = (\lambda_i\delta_{ij})_{i,j=0,1,\cdots,m}
$$

and

$$
G(Y) = \Big((g'(\sum_{j=1}^{m}\alpha_j e_j)\sum_{j=1}^{m}\alpha_j\nabla e_j, \nabla e_1), \cdots, (g'(\sum_{j=1}^{m}\alpha_j e_j)\sum_{j=1}^{m}\alpha_j\nabla e_j, \nabla e_m)\Big).
$$

The matrix M is positive definite and $G(Y)$ depends continuously on Y. Hence (2.2.6) or (2.2.3) has a unique solution on some finite time interval $[0, T_m)$.

Next, we will show that the solutions are bounded in time and uniformly bounded in m. The system (2.2.3) is equivalent to

$$\begin{aligned} &\tau\frac{d\phi_m}{dt} - \Delta\left(\delta\frac{d\phi_m}{dt} - \Delta\phi_m + P_m g(\phi_m) - u_m\right) = 0, \\ &\epsilon\frac{du_m}{dt} + \frac{d\phi_m}{dt} - \Delta u_m = 0. \end{aligned} \tag{2.2.7}$$

Taking note that $\|u_m(0)\| \le \|u_0\|$ and $\|\phi_m(0)\| \le \|\phi_0\|$, ϕ_m and u_m satisfy the *a priori estimates* in section 2.1. Thus replacing ϕ and u with ϕ_m and u_m, integrating (2.1.15) from 0 to t, $0 < t < T$, and making use of (2.1.12) we deduce that

$$\sup_{t\in[0,T]} \|\phi_m(t)\|_1^2 \le C \quad \text{and} \quad \sup_{t\in[0,T]} \varepsilon\|u_m(t)\|_1^2 \le C,$$

where C is independent of t and m.

Thus the sequences ϕ_m and u_m remains uniformly bounded in $L^\infty(0, T; V_1)$ and $L^\infty(0, T; L^2(\Omega))$ respectively.

Next, we again deduce from (2.1.15) that (by integration from 0 to T)

$$\int_0^T \left(\|u_m\|_1^2 + \|\phi_m\|_2^2 + \|\phi_{mt}\|_{-1}^2 + \delta\|\phi_{mt}\|^2\right) dt \le CT. \tag{2.2.8}$$

The right hand side in (2.2.8) is independent on m, therefore, the sequences ϕ_m, u_m and $\dfrac{d\phi_m}{dt}$ remains uniformly bounded in $L^2(0, T; V_2)$, $L^2(0, T; V_1)$, and $L^2(0, T; L^2(\Omega))$ respectively, then from $(2.2.7)_2$ we get that $\varepsilon\dfrac{du_m}{dt}$ remains uniformly bounded in $L^2(0, T; V_{-1})$ as well (with a bound independent of ε).

Passage to the limit:

We can extract subsequences still denoted as ϕ_m and u_m such that

$$\begin{aligned}
&\phi_m \rightharpoonup \phi \ \text{ in } \ L^\infty(0,T;V_1) \ \text{ weakly} - \text{star}, \\
&\phi_m \rightharpoonup \phi \ \text{ in } \ L^2(0,T;V_2) \ \text{ weakly}, \\
&\sqrt{\varepsilon}u_m \rightharpoonup \sqrt{\varepsilon}u \ \text{ in } \ L^\infty(0,T;L^2(\Omega)) \ \text{ weakly} - \text{star}, \\
&u_m \rightharpoonup u \ \text{ in } \ L^2(0,T;V_1) \ \text{ weakly}, \\
&\sqrt{\delta}\phi_{mt} \rightharpoonup \sqrt{\delta}\phi_t \ \text{ in } \ L^2(0,T;L^2(\Omega)) \ \text{ weakly}, \\
&\phi_{mt} \rightharpoonup \phi_t \ \text{ in } \ L^2(0,T;V_{-1}) \ \text{ weakly}.
\end{aligned} \tag{2.2.9}$$

Now, we show convergence for the nonlinear term $g(\phi_m)$. From (1.5.6), and due to ϕ_m uniformly bounded in $L^\infty(0,T;V_1)$, we have

$$\begin{aligned}
\|g(\phi_m)\|^2_{L^2(\Omega\times(0,T))} &\leq c\int_0^T \int_\Omega (1+|\phi_m|^{p+2})dxdt \\
&\leq c\int_0^T (1+\|\phi_m\|_1^{p+2})dt \\
&\leq C.
\end{aligned} \tag{2.2.10}$$

Thus, up to a subsequence, $g(\phi_m) \rightharpoonup \psi$ in $L^2(\Omega\times(0,T))$.

From (2.2.9), due to Lemma 1.1, we deduce that $\phi_m \to \phi$ in $L^2(\Omega\times(0,T))$ and a.e. (x,t) in $\Omega\times(0,T)$. It follows from the continuity of g that $g(\phi_m) \to g(\phi)$ a.e. (x,t) in $\Omega\times(0,T)$. Hence, by the dominated convergence theorem (cf. [68], Lemma 8.3) $g(\phi_m) \rightharpoonup g(\phi)$ in $L^2(\Omega\times(0,T))$. Then, the uniqueness of weak limits gives that $g(\phi) = \psi$.

Now, in other to pass to the limit in the approximate problem (2.2.3), we consider a scalar function Ψ and Φ continuously differentiable on $[0,T]$ and such that $\Psi(T)=\Phi(T)=0$.

Next, we multiply $(2.2.3)_1$ by $\Psi(t)$ and $(2.2.3)_2$ by Φ, integrate with respect to t and integrate by parts, we get respectively;

$$
\begin{aligned}
&-\tau\int_0^T(\phi_m,e_j)\frac{d\Psi}{dt}(t)dt-\delta\int_0^T(\phi_m,-\Delta e_j)\frac{d\Psi}{dt}(t)dt+\int_0^T(\Delta\phi_m,\Delta e_j)\Psi(t)dt\\
&+\int_0^T(g(\phi_m),-\Delta e_j)\Psi(t)dt-\int_0^T(\nabla u_m,\nabla e_j)\Psi(t)dt\\
&=(\tau+\delta)(\phi_{0m},e_j)\Psi(0), \qquad (2.2.11)\\
&-\varepsilon\int_0^T(u_m,e_j)\frac{d\Phi}{dt}(t)dt-\int_0^T(\phi_m,e_j)\frac{d\Phi}{dt}(t)dt+\int_0^T(\nabla u_m,\nabla e_j)\Phi(t)dt\\
&=\varepsilon(u_{0m},e_j)\Phi(0)+(\phi_{0m},e_j)\Phi(0). \qquad (2.2.12)
\end{aligned}
$$

Passing to the limit as $m\to\infty$, and taking into account (2.2.4), we deduce that (2.2.11) and (2.2.12) hold, for ϕ_m and u_m replaced with ϕ and u respectively, for each $j=0,1,\cdots,m$. Therefore,

$$
\begin{aligned}
&-\tau\int_0^T(\phi,q)\frac{d\Psi}{dt}(t)dt-\delta\int_0^T(\nabla\phi,\nabla q)\frac{d\Psi}{dt}(t)dt+\int_0^T(\Delta\phi,\Delta q)\Psi(t)dt\\
&+\int_0^T(\nabla g(\phi),\nabla q)\Psi(t)dt-\int_0^T(\nabla u,\nabla q)\Psi(t)dt\\
&=(\tau+\delta)(\phi_0,q)\Psi(0),\quad \forall q\in V_2, \qquad (2.2.13)
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon\int_0^T(u,v)\frac{d\Phi}{dt}(t)dt-\int_0^T(\phi,v)\frac{d\Phi}{dt}(t)dt+\int_0^T(\nabla u,\nabla v)\Phi(t)dt\\
&=\varepsilon(u_0,v)\Phi(0)+(\phi_0,v)\Phi(0),\quad \forall v\in V_1. \qquad (2.2.14)
\end{aligned}
$$

Now, considering in particular, for any $\zeta, \chi \in \mathcal{D}(0,T)$ with $\Psi = \zeta, \chi = \Phi$, it follows from (2.2.13) and (2.2.14) that the following inequalities hold in the sense of distribution on $(0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\big[\tau(\phi,q)+\delta(\nabla\phi,\nabla q)\big]+(\Delta\phi,\Delta q)+(\nabla g(\phi),\nabla q)-(\nabla u,\nabla q)=0, \quad \forall q\in V_2,\\
&\frac{d}{dt}\big[\varepsilon(u,v)+(\phi,v)\big]+(\nabla u,\nabla v)=0, \quad \forall v\in V_1,
\end{aligned} \tag{2.2.15}$$

which is exactly (2.2.2). Thus the following equalities hold

$$\tau\phi_t-\delta\Delta\phi_t+\Delta^2\phi,-\Delta g(\phi)+\Delta u=0, \quad \text{in} \quad L^2(0,T;V_{-2}), \tag{2.2.16}$$

$$\varepsilon u_t+\phi_t-\Delta u=0, \quad \text{in} \quad L^2(0,T;V_{-1}). \tag{2.2.17}$$

Finally, we need to show that $\phi(0)=\phi_0$ and $u(0)=u_0$. However, we will show first that ϕ and u are continuous on $[0,T]$. From (2.2.17), we get $\sqrt{\varepsilon}u_t \in L^2(0,T;V_{-1})$. From classical compactness theorems (cf. Lemma 1.2), it follows that ϕ is weakly continuous from $[0,T]$ into V_1, and $\sqrt{\varepsilon}u$ is weakly continuous from $[0,T]$ into $L^2(\Omega)$. Using (2.1.1) and due to (1.5.6), we can see that

$$|V(t)-V(t')|\le c\int_{t'}^{t}(\|\phi_t(s)\|^2+\|\nabla u(s)\|^2+\|\phi(s)\|_1^{2p+2}+1)ds.$$

where $V(t)=\|\phi(t)\|_1^2+\varepsilon\|u(t)\|^2$. Hence, as $t\to t'$, we deduce that the real function $t\to\|\phi(t)\|_1^2+\epsilon\|u(t)\|^2$ is continuous on $[0,T]$. We can conclude that

$$\phi\in\mathcal{C}([0,T];V_1) \quad \text{and} \quad \sqrt{\epsilon}u\in\mathcal{C}([0,T];L^2(\Omega)). \tag{2.2.18}$$

Now, we multiply $(2.2.15)_1$ by $\Psi(t)$ and $(2.2.15)_2$ by $\Phi(t)$ (Ψ and Φ as before), integrate with respect to t and integrate by parts, we get

$$
\begin{aligned}
&- \tau \int_0^T (\phi, q)\Psi'(t)dt - \delta \int_0^T (\nabla\phi, \nabla q)\Psi'(t)dt + \int_0^T (\Delta\phi, \Delta q)\Psi(t)dt \\
&\quad + \int_0^T (\nabla g(\phi), \nabla q)\Psi(t)dt - \int_0^T (\nabla u, \nabla q)\Psi(t)dt \\
&= (\tau + \delta)(\phi(0), q)\Psi(0), \quad \forall q \in V_2,
\end{aligned}
\tag{2.2.19}
$$

$$
\begin{aligned}
&- \varepsilon \int_0^T (u, v)\Phi'(t)dt + \int_0^T (\phi, v)\Phi'(t)dt + \int_0^T (\nabla u, \nabla v)\Phi(t)dt \\
&= \varepsilon(u(0), v)\Phi(0) + (\phi(0), v)\Phi(0), \quad \forall v \in V_1.
\end{aligned}
\tag{2.2.20}
$$

From (2.2.13) and (2.2.19), we deduce

$$
(\tau + \delta)(\phi(0) - \phi_0, q)\Psi(0) = 0, \quad \forall q \in V_2, \tag{2.2.21}
$$

as well, from (2.2.14) and (2.2.20), we deduce

$$
\varepsilon(u(0) - u_0, v)\Phi(0) + (\phi(0) - \phi_0, v)\Phi(0) = 0, \quad \forall v \in V_1, \tag{2.2.22}
$$

for every Ψ, Φ such that $\Psi(T) = \Phi(T) = 0$. Hence, we can choose Ψ, Φ such that $\Psi(0) \neq 0$ and $\Phi(0) \neq 0$. Therefore, from (2.2.21) we have

$$
(\tau + \delta)(\phi(0) - \phi_0, q) = 0, \quad \forall q \in V_2,
$$

which implies that $\phi(0) = \phi_0$, it then follows from (2.2.22) that

$$
\varepsilon(u(0) - u_0, v) = 0, \quad \forall v \in V_1,
$$

which implies that $u(0) = u_0$.

(ii) Uniqueness: Let (ϕ_1, u_1) and (ϕ_2, u_2) be two solutions of 2.0.1. Setting $\phi = \phi_1 - \phi_2$ and $u = u_1 - u_2$, we have $\phi(0) = 0$, $u(0) = 0$, $m(\phi(t)) = 0$, $m(u(t)) = 0$, $\forall t \geq 0$, and (ϕ, u) satisfies the equations

$$\tau\phi_t + N(\delta\phi_t + N\phi + g(\phi_1) - g(\phi_2) - u) = 0, \tag{2.2.23}$$

$$\varepsilon u_t + \phi_t + Nu = 0. \tag{2.2.24}$$

We multiply (2.2.23) by $N^{-1}\phi_t$, and (2.2.24) by u and we integrate over Ω, then summing the resulting equation, we reach

$$\frac{1}{2}\frac{d}{dt}\left(\|\phi\|_1^2 + \varepsilon\|u\|^2\right) + \|u\|_1^2 + \tau\|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2 \le |(g(\phi_1) - g(\phi_2), \phi_t)|. \tag{2.2.25}$$

Using (1.5.6), we have the following estimates.

When $d = 1$, we have (cf. (2.1.20))

$$\|g(\phi_1) - g(\phi_2)\|^2 \le \sup_{\theta\in[0,1]} \|g'(\theta\phi_1 + (1-\theta)\phi_2)\|_{L^\infty(\Omega)}^2 \|\phi\|^2. \tag{2.2.26}$$

When $d = 2$, we have

$$\begin{aligned}\|g(\phi_1) - g(\phi_2)\|^2 &\le c(1 + \|\phi_1\|_{L^{3p+3}(\Omega)}^{2p+2} + \|\phi_2\|_{L^{3p+3}(\Omega)}^{2p+2})\|\phi\|_{L^6(\Omega)}^2 \\ &\le c(1 + \|\phi_1\|_1^{2p+2} + \|\phi_2\|_1^{2p+2})\|\phi\|_1^2. \end{aligned}\tag{2.2.27}$$

When $d = 3$, using Agmon's inequality, we have

$$\begin{aligned}&\|g(\phi_1) - g(\phi_2)\|^2 \\ &\le c\int_\Omega |\phi|^2 + c\int_\Omega (|\phi_1|^8 + |\phi_2|^8)|\phi|^2 \\ &\le c\|\phi\|^2 + c(\|\phi_1\|_{L^\infty(\Omega)}^4 + \|\phi_2\|_{L^\infty(\Omega)}^4)(\|\phi_1\|_{L^6(\Omega)}^4 + \|\phi_2\|_{L^6(\Omega)}^4)\|\phi\|_{L^6(\Omega)}^2 \\ &\le c\|\phi\|^2 + c(\|\phi_1\|_1^2 + \|\phi_2\|_1^2)(\|\phi_1\|_2^2 + \|\phi_2\|_2^2)(\|\phi_1\|_1^4 + \|\phi_2\|_1^4)\|\phi\|_1^2 \\ &\le c(1 + \|\phi_1\|_1^6 + \|\phi_2\|_1^6)(1 + \|\phi_1\|_2^2 + \|\phi_2\|_2^2)\|\phi\|_1^2. \end{aligned}\tag{2.2.28}$$

Therefore, from (2.2.25) and application of Young's inequality, we deduce that

$$\frac{d}{dt}\left(\|\phi\|_1^2+\varepsilon\|u\|^2\right)+2\|u\|_1^2+2\tau\|\phi_t\|_{-1}^2+\delta\|\phi_t\|^2\le\frac{M_2(t)}{\delta}\|\phi\|_1^2, \tag{2.2.29}$$

where

$$M_2(t)=\begin{cases} c\sup\limits_{\theta\in[0,1]}\|g'(\theta\phi_1+(1-\theta)\phi_2)\|_{L^\infty(\Omega)}^2, & \text{if } \ d=1,\\ c(1+\|\phi_1\|_1^{2p+2}+\|\phi_2\|_1^{2p+2}), & \text{if } \ d=2,\\ c(1+\|\phi_1\|_1^6+\|\phi_2\|_1^6)(1+\|\phi_1\|_2^2+\|\phi_2\|_2^2), & \text{if } \ d=3.\end{cases}$$

Observe that from (2.1.15), we have that $\phi_1,\phi_2\in L^2(0,T;V_2)$ so that $M_2(t)\in L^1(0,T)$. Applying the Gronwall's lemma to (2.2.29), we deduce that

$$\|(\phi(t),u(t))\|_{\mathcal{U}_{1,\varepsilon}}^2\le ce^{\frac{1}{\delta}\int_0^t M_2(s)ds}\|(\phi(0),u(0))\|_{\mathcal{U}_{1,\varepsilon}}^2,\ \forall t\ge 0, \tag{2.2.30}$$

hence the result.

(iii) If $(\phi_0,u_0)\in\mathcal{U}_2$, we can proceed like in part (i) to show the existence of a pair of functions (ϕ,u) solution to (2.0.1) such that $\phi\in\mathcal{C}([0,T];V_2)\cap L^2(0,T;V_3)$ and $u\in\mathcal{C}([0,T];V_1)\cap L^2(0,T;V_2)$.

Thus the proof of Theorem (2.1) is complete. ∎

2.3 The global attractor

Thanks to Theorem 2.1, we can define the semigroup

$$S_\varepsilon(t):\mathcal{U}_1\to\mathcal{U}_1,\quad(\phi_0,u_0)\mapsto(\phi(t),u(t)),\quad t\ge 0,$$

where $(\phi(t), u(t))$ is the solution to (2.0.1) at time t. From Theorem 2.1, we deduce that the semigroup $S_\varepsilon(t)$ is strongly continuous. We apply Gronwall's lemma to (2.1.15) and we deduce the existence of an absorbing set for $S_\varepsilon(t)$ on $K_{\alpha,\sigma}$ of the form

$$\mathcal{B}_1 = \{(\varphi, \psi) \in K_{\alpha,\sigma},\ \|(\varphi, \psi)\|_{\mathcal{U}_{1,\varepsilon}} \le r_1\},$$

where r_1 is independent of ε. Note that, if $(\phi, u) \in K_{\alpha,\sigma}$, then the constant c_0 in (2.1.15) is bounded from below by a strictly positive constant that does not depend on $m(\phi_0)$ and $m(u_0)$, the other constants are also independent of $m(\phi_0)$ and $m(u_0)$. Now, let (ϕ_0, u_0) be in a bounded set B of $K_{\alpha,\sigma}$, then there exists $t_1 > 0$ depending only on B such that $\|(\phi(t), u(t))\|_{\mathcal{U}_{1,\varepsilon}} \le r_1,\ \forall t \ge t_1$. We also deduce from (2.1.15) that $\int_t^{t+1} \|(\phi(s), u(s))\|^2_{\mathcal{U}_{2,\varepsilon}} ds \le c,\ \forall t \ge t_1$. Applying the uniform Gronwall lemma to (2.1.23), we deduce the existence of an absorbing set for $S_\varepsilon(t)$ on $K_{\alpha,\sigma}$ of the form

$$\mathcal{B}_2 = \{(\varphi, \psi) \in K_{\alpha,\sigma} \cap \mathcal{U}_2,\ \|(\varphi, \psi)\|_{\mathcal{U}_{2,\varepsilon}} \le r_{\delta,2}\},$$

where $r_{\delta,2}$ is independent of ε.

Proposition 2.1 *The semigroup $S_\varepsilon(t)$ restricted to $K_{\alpha,\sigma}$ is uniformly compact for every $\varepsilon \in (0, 1]$.*

Proof. Let $B \subset K_{\alpha,\sigma}$ be a bounded set. Then there exists $t(B) \geqslant 0$ independent of ε, s.t $\bigcup\limits_{t \geqslant t(B)+1} S_\varepsilon(t)B$ is bounded in $K_{\alpha,\sigma}$ and in $K_{\alpha,\sigma} \cap \mathcal{U}_2$. Since $\mathcal{U}_2$ is compact in $\mathcal{U}_1$, we have that $\bigcup\limits_{t \geqslant t(B)+1} S_\varepsilon(t)B$ is precompact in $\mathcal{U}_1$, i.e., $\overline{\bigcup\limits_{t \geqslant t(B)+1} S_\varepsilon(t)B}$ is compact in $\mathcal{U}_1$. ∎

We apply Theorem 1.1 and we obtain the following result.

Theorem 2.2 *For every $\varepsilon \in (0,1]$, the semigroup $S_\varepsilon(t)$ has the global attractor $\mathcal{A}_\varepsilon^{\alpha,\sigma}$ in $K_{\alpha,\sigma}$.*

The semigroup $S(t)$ generated by the unperturbed problem (the viscous Cahn-Hilliard equation (1.2.7)) possesses the global attractor $\mathcal{A}^\alpha$ on K_α (see [7]). Observe that a solution of the unperturbed problem for both variables ϕ and u (at time t) is given by

$$\phi(t) = S(t)\phi_0 \quad \text{and} \quad u(t) = \mathrm{L}_{m(u_0)}\phi(t),$$

where

$$\mathrm{L}_\beta\varphi = \beta + [(1+\tau)I + \delta N]^{-1}(N\varphi + \overline{g(\varphi)}). \tag{2.3.1}$$

We define

$$(\mathcal{A}^\alpha)^\sigma = \bigcup_{|\beta|\leq\sigma} \{(\varphi, \mathrm{L}_\beta\varphi), \ \ \varphi \in \mathcal{A}^\alpha\}.$$

We now give the following stability property, whose proof is given in Theorem 3.2 of the next Chapter.

Theorem 2.3 *The global attractor $\mathcal{A}_\varepsilon^{\alpha,\sigma}$ is upper semicontinuous at $\varepsilon = 0$, that is,*

$$\lim_{\varepsilon\to 0} \mathrm{dist}_{\mathcal{U}_2}(\mathcal{A}_\varepsilon^{\alpha,\sigma}, (\mathcal{A}^\alpha)^\sigma) = 0. \tag{2.3.2}$$

2.4 Inertial manifolds

In this section, we take $d = 1$ or 2, and we assume $\Omega = \Pi_{i=1}^d (0, L_i)$ and L_1/L_2 is a rational number. Also, we consider Neuman (1.5.1) or periodic (1.5.2) boundary conditions when $d = 1$ and only the periodic (1.5.2) boundary condition when $d = 2$. In order to prove the existence of an inertial manifold for Problem (2.0.1), we introduce the "prepared problem":

$$\begin{cases} \tau\phi_t + N(\delta\phi_t + N\phi + \mathfrak{g}(\phi) - u) = 0, \\ \\ \varepsilon u_t + \phi_t + Nu = 0, \end{cases} \tag{2.4.1}$$

where
$$\mathfrak{g}(\phi) = \theta\left(\frac{\|\Lambda\phi\|}{r_d}\right) g(\phi), \tag{2.4.2}$$

r_d is the radius of the absorbing set $\mathcal{B}_d \subset K_{\alpha,\sigma} \cap \mathcal{U}_d$ for the semigroup $S_\varepsilon(t)|_{K_{\alpha,\sigma}}$, $d = 1, 2$,

$$\Lambda = \begin{cases} (I+N)^{1/2}, & \text{if} \quad d = 1, \\ \\ I + N, & \text{if} \quad d = 2, \end{cases} \tag{2.4.3}$$

and $\theta : \mathbb{R}^+ \to [0, 1]$ is a $\mathcal{C}^1$ function such that $\theta(s)$ is equal to 1 when $0 \le s \le 1$, and is equal to 0 when $s > 2$, and $|\theta'(s)| \le 2$, $\forall s \ge 0$.

Then we write (2.4.1) in the following form:

$$U_t + \mathcal{A}U + \mathcal{G}(U) = 0, \tag{2.4.4}$$

where $U = (\phi, u)$,

$$\mathcal{G}(U) = \left((\tau I + \delta N)^{-1} N\mathfrak{g}(\phi), -\frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N\mathfrak{g}(\phi)\right),$$

and
$$\mathcal{A} = \begin{pmatrix} (\tau I + \delta N)^{-1} N^2 & -(\tau I + \delta N)^{-1} N \\ -\frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N^2 & \frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N + \frac{1}{\varepsilon} N \end{pmatrix}.$$

The operator $\mathcal{A} : \mathcal{U}_3 \to \mathcal{U}_1$ has positive eigenvalues

$$\mu_k^{\pm} = \frac{\lambda_k}{2\varepsilon(\tau+\delta\lambda_k)} \left(1+\tau+(\delta+\epsilon)\lambda_k \pm \sqrt{(1+\tau+(\delta+\varepsilon)\lambda_k)^2 - 4\varepsilon\lambda_k(\tau+\delta\lambda_k)}\right),$$

for $k = 0, 1, 2, ...$ and corresponding eigenfunctions $U_k^{\pm} = (e_k, -\tilde{\mu}_k^{\pm} e_k)$,

where $\tilde{\mu}_k^{\pm} = \dfrac{1}{2\varepsilon} \left(1+\tau+(\delta-\varepsilon)\lambda_k \pm \sqrt{(1+\tau+(\delta+\varepsilon)\lambda_k)^2 - 4\varepsilon\lambda_k(\tau+\delta\lambda_k)}\right)$,

$\{\lambda_k\}$ are the eigenvalues of N ordered in an increasing sequence and $\{e_k\}$ are the corresponding eigenfunctions, see (1.5.11)-(1.5.13).

Proposition 2.2 *The operator $\mathcal{A}$ is non self-adjoint.*

Proof. Let $U = (u_1, u_2),\ V = (v_1, v_2) \in \mathcal{U}_3$ and $\langle \cdot, \cdot \rangle$ be the inner product defined in (2.4.16).

We have that

$$\mathcal{A}U = \begin{pmatrix} (\tau I + \delta N)^{-1} N^2 u_1 - (\tau I + \delta N)^{-1} N u_2 \\ \\ -\frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N^2 u_1 + \frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N u_2 + \frac{1}{\varepsilon} N u_2 \end{pmatrix}$$

and

$$\mathcal{A}V = \begin{pmatrix} (\tau I + \delta N)^{-1} N^2 v_1 - (\tau I + \delta N)^{-1} N v_2 \\ \\ -\frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N^2 v_1 + \frac{1}{\varepsilon}(\tau I + \delta N)^{-1} N v_2 + \frac{1}{\varepsilon} N v_2 \end{pmatrix}$$

Hence

$$
\begin{aligned}
&\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle \\
=&(1+\tau)\big[-((\tau I+\delta N)^{-1}Nu_2, v_1) + (u_1, (\tau I+\delta N)^{-1}Nv_2)\big] \\
&+\delta\big[-((\tau I+\delta N)^{-1}N\nabla u_2, \nabla v_1) + (\nabla u_1, (\tau I+\delta N)^{-1}N\nabla v_2)\big] \\
&+(v_1, (\tau I+\delta N)^{-1}N\nabla u_2) + (v_1, Nu_2) + \varepsilon(Nu_2, v_2) \\
&-(u_1, (\tau I+\delta N)^{-1}N\nabla v_2) - (u_1, Nv_2) - \varepsilon(u_2, Nv_2) \\
=&\tau\big[(u_1, (\tau I+\delta N)^{-1}Nv_2) - ((\tau I+\delta N)^{-1}Nu_2, v_1)\big] \\
&+\delta\big[-((\tau I+\delta N)^{-1}N\nabla u_2, \nabla v_1) + (\nabla u_1, (\tau I+\delta N)^{-1}N\nabla v_2)\big] \\
&+(v_1, Nu_2) - (u_1, Nv_2).
\end{aligned}
$$

In particular, let $u_1 = u_2 = v_1$ and $v_2 = 2u_1$, we can choose $u_1 \in V_3 = \mathscr{D}(N^{3/2})$ different from a constant such that

$$
\begin{aligned}
\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle &= \tau\|(\tau I+\delta N)^{-1/2}\nabla u_1\|^2 + \delta\|(\tau I+\delta N)^{-1/2}Nu_1\|^2 - \|\nabla u_1\|^2 \\
&\neq 0.
\end{aligned}
$$

Therefore the operator $\mathcal{A}$ is not symmetric, hence $\mathcal{A}$ is non self-adjoint. ∎

Let $c_1 > 0$. There exists n such that $\lambda_n \geq 1$ and

$$
\lambda_{n+1} - \lambda_n > \max\{4c_1(\tau+1), 4c_1\delta\}. \tag{2.4.5}
$$

Indeed, if $d = 1$, then this is immediate; and if $d = 2$, then the result is due to I. Richards (see [67]), since L_1/L_2 is rational. Of course, the latter estimate implies that

$$
\lambda_{n+1}^2 - \lambda_n^2 > \max\{4c_1(\tau+1), 4c_1\delta\}(\lambda_n + \lambda_{n+1}).
$$

Denote

$$
\begin{aligned}
D_k &= \varepsilon^2\lambda_k^2 + 2\varepsilon(1-\tau-\delta\lambda_k)\lambda_k + (1+\tau+\delta\lambda_k)^2,\\
\Delta_k &= \frac{\lambda_{k+1}\sqrt{D_{k+1}}}{\tau+\delta\lambda_{k+1}} - \frac{\lambda_k\sqrt{D_k}}{\tau+\delta\lambda_k},\\
f_k &= \frac{\lambda_{k+1}^2}{\tau+\delta\lambda_{k+1}} - \frac{\lambda_k^2}{\tau+\delta\lambda_k} - 2c_1,\\
g_k^{\pm} &= \frac{(1+\tau+\delta\lambda_{k+1})\lambda_{k+1}}{\tau+\delta\lambda_{k+1}} \pm \frac{(1+\tau+\delta\lambda_k)\lambda_k}{\tau+\delta\lambda_k},\\
h_k^{\pm} &= \frac{\lambda_{k+1}^4}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_k^4}{(\tau+\delta\lambda_k)^2},\\
i_k^{\pm} &= \frac{(1-\tau-\delta\lambda_{k+1})\lambda_{k+1}^3}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{(1-\tau-\delta\lambda_k)\lambda_k^3}{(\tau+\delta\lambda_k)^2},\\
j_k^{\pm} &= \frac{(1+\tau+\delta\lambda_{k+1})^2\lambda_{k+1}^2}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{(1+\tau+\delta\lambda_k)^2\lambda_k^2}{(\tau+\delta\lambda_k)^2}.
\end{aligned}
$$

Let us now prove the following results.

Lemma 2.1 *Provided that n is large enough for* (2.4.5) *to hold, there exists $\tilde{\varepsilon}(n)$ suitably small such that the following inequalities are satisfied:*

(i) $f_n \geq 0$.

(ii) $\Delta_n \geq 0, \quad \text{for every} \quad \varepsilon \in (0, \tilde{\varepsilon}(\mathrm{n})]$.

(iii) $f_n^4 - 2f_n^2 h_n^+ + (h_n^-)^2 > 0$.

(iv) $i_n^- g_n^+ + f_n(g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n) < 0$.

(v) $\varepsilon^2 h_n^+ + 2\varepsilon i_n^+ + j_n^+ - (\varepsilon f_n + g_n^-)^2 > 0, \quad \text{for every} \quad \varepsilon \in (0, \tilde{\varepsilon}(\mathrm{n})]$.

Proof. $\tilde{\varepsilon}(n) \in (0,1]$ and may change from one line to the other.

(i) $f_n = \dfrac{(\tau+\delta\lambda_n)\lambda_{n+1}^2 - (\tau+\delta\lambda_{n+1})\lambda_n^2 - 2c_1(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})}$.

We have

$$(\tau+\delta\lambda_n)\lambda_{n+1}^2-(\tau+\delta\lambda_{n+1})\lambda_n^2-2c_1(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})$$

$$=\tau(\lambda_{n+1}^2-\lambda_n^2)+\delta\lambda_n\lambda_{n+1}(\lambda_{n+1}-\lambda_n)-2c_1\tau^2-2c_1\tau\delta(\lambda_{n+1}+\lambda_n)-2c_1\delta^2\lambda_n\lambda_{n+1}$$

$$=2\tau(\lambda_{n+1}+\lambda_n)(\lambda_{n+1}-\lambda_n-4c_1\tau\delta)+2\tau(\lambda_{n+1}^2-\lambda_n^2-4c_1\tau)$$

$$+\delta\lambda_n\lambda_{n+1}(\lambda_{n+1}-\lambda_n-2c_1\delta)>0$$

holds true, whenever (2.4.5) is satisfied. Hence $f_n \geq 0$.

(ii) The inequality $\Delta_n \geq 0$ is equivalent to

$$\tau\left(\lambda_{n+1}\sqrt{D_{n+1}}-\lambda_n\sqrt{D_n}\right)+\delta\lambda_n\lambda_{n+1}\left(\sqrt{D_{n+1}}-\sqrt{D_n}\right)\geq 0. \tag{2.4.6}$$

Now, the terms $\lambda_{n+1}\sqrt{D_{n+1}}-\lambda_n\sqrt{D_n}$ and $\lambda_{n+1}^2D_{n+1}-\lambda_n^2D_n$ have the same sign, and

$$\begin{aligned}\lambda_{n+1}^2D_{n+1}-\lambda_n^2D_n =&(\delta-\varepsilon)^2(\lambda_{n+1}^4-\lambda_n^4)+2\left[\delta(1+\tau)+\varepsilon(1-\tau)\right](\lambda_{n+1}^3-\lambda_n^3)\\ &+(1+\tau)^2(\lambda_{n+1}^2-\lambda_n^2),\end{aligned} \tag{2.4.7}$$

which is a positive quantity for every $\varepsilon \in (0, \varepsilon(n)]$ and for some $\varepsilon(n) > 0$.

Similarly, the terms $\sqrt{D_{n+1}}-\sqrt{D_n}$ and $D_{n+1}-D_n$ have the same sign, and

$$D_{n+1}-D_n =(\delta-\varepsilon)^2(\lambda_{n+1}^2-\lambda_n^2)+2\left[\delta(1+\tau)+\varepsilon(1-\tau)\right](\lambda_{n+1}-\lambda_n), \tag{2.4.8}$$

It is clear that both quantities (2.4.7) and (2.4.8) are positive for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, for some $\tilde{\varepsilon}(n) > 0$, hence the result.

(iii) The quadratic equation $x^2-2h_n^+x+(h_n^-)^2=0$ has two positive real roots

$$x^{\pm}=\left(\frac{\lambda_{n+1}^2}{\tau+\delta\lambda_{n+1}}\pm\frac{\lambda_n^2}{\tau+\delta\lambda_n}\right)^2,$$

and we have $f_n^2 < x^- \le x^+$, hence (iii).

(iv) A computation shows that

$$\begin{aligned}
&i_n^- g_n^+ - i_n^+ g_n^- \\
=&\frac{2\lambda_n\lambda_{n+1}}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})} \\
&\times\left[\frac{(1+\tau+\delta\lambda_n)(1-\tau-\delta\lambda_{n+1})\lambda_{n+1}^2}{\tau+\delta\lambda_{n+1}}-\frac{(1-\tau-\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_n^2}{\tau+\delta\lambda_n}\right]
\end{aligned} \tag{2.4.9}$$

and

$$\begin{aligned}
&f_n\left[(g_n^-)^2 - j_n^+\right] \\
&= -2\lambda_n\lambda_{n+1}\frac{(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})}\left(\frac{\lambda_{n+1}^2}{\tau+\delta\lambda_{n+1}}-\frac{\lambda_n^2}{\tau+\delta\lambda_n}-2c_1\right).
\end{aligned} \tag{2.4.10}$$

Summing (2.4.9) with (2.4.10), we obtain

$$\begin{aligned}
&i_n^- g_n^+ + f_n(g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n) \\
&= \Big((1+\tau+\delta\lambda_n)\lambda_{n+1}^2 - (1+\tau+\delta\lambda_{n+1})\lambda_n^2 - c_1(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\Big) \\
&\quad\times -\frac{4\lambda_n\lambda_{n+1}}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})}.
\end{aligned}$$

Exactly like in (i), we have that

$$\begin{aligned}
&(1+\tau+\delta\lambda_n)\lambda_{n+1}^2 - (1+\tau+\delta\lambda_{n+1})\lambda_n^2 - c_1(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1}) \\
&= \frac{1+\tau}{2}(\lambda_{n+1}+\lambda_n)(\lambda_{n+1}-\lambda_n-2c_1\delta) + \frac{1+\tau}{2}(\lambda_{n+1}^2-\lambda_n^2-2c_1(1+\tau)) \\
&+ \delta\lambda_n\lambda_{n+1}(\lambda_{n+1}-\lambda_n-c_1\delta) > 0
\end{aligned}$$

holds whenever (2.4.5) is satisfied. Hence (iv) holds.

(v) The inequality

$$\varepsilon^2 h_n^+ + 2\varepsilon i_n^+ + j_n^+ - \left(\varepsilon f_n + g_n^-\right)^2 > 0$$

is equivalent to

$$\varepsilon^2\left(h_n^+ - f_n^2\right) + 2\varepsilon\left(i_n^+ - f_n g_n^-\right) + j_n^+ - (g_n^-)^2 > 0.$$

A computation shows that

$$h_n^+ - f_n^2 = \frac{2\lambda_n^2\lambda_{n+1}^2}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})} + 4c_1\left(\frac{\lambda_{n+1}^2}{\tau+\delta\lambda_{n+1}} - \frac{\lambda_n^2}{\tau+\delta\lambda_n} - c_1\right) > 0,$$

and

$$j_n^+ - (g_n^-)^2 = 2\frac{(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_n\lambda_{n+1}}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})} > 0.$$

Hence (v) holds for all $\varepsilon \in (0, \tilde{\varepsilon}(n)]$. The proof of the lemma is completed. ∎

We have the following result.

Proposition 2.3 *For every $c_1 > 0$, there exists n (independent of ε) and a $\tilde{\varepsilon}(n) > 0$ such that, for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, the spectral gap condition holds:*

$$\mu_{n+1}^- - \mu_n^- > c_1. \tag{2.4.11}$$

Proof. We observe that for every $k \geq 0$,

$$g_k^- = \frac{(\lambda_{k+1}-\lambda_k)\left[\tau(1+\tau) + \delta(\lambda_{k+1}+\lambda_k) + \delta^2\lambda_k\lambda_{k+1}\right]}{(\tau+\delta\lambda_k)(\tau+\delta\lambda_{k+1})} \geq 0.$$

We have

$$\mu_{n+1}^- - \mu_n^- = \frac{1}{2\varepsilon}(\varepsilon f_n + g_n^- - \Delta_n) + c_1.$$

Thanks to (i) and (ii) of Lemma 2.1, the inequality $\varepsilon f_n + g_n^- - \Delta_n > 0$ is equivalent to

$$\left(\varepsilon f_n + g_n^-\right)^2 > \Delta_n^2. \tag{2.4.12}$$

We set $\Delta_n = A_{n+1} - A_n$, with $A_n = \dfrac{\lambda_n\sqrt{D_n}}{\tau + \delta\lambda_n}$, so that (2.4.12) is exactly the following

$$(\varepsilon f_n + g_n^-)^2 > A_{n+1}^2 + A_n^2 - 2A_{n+1}A_n. \tag{2.4.13}$$

We observe that

$$A_{n+1}^2 + A_n^2 = \varepsilon^2 h_n^+ + 2\varepsilon i_n^+ + j_n^+ \quad \text{and} \quad A_{n+1}^2 - A_n^2 = \varepsilon^2 h_n^- + 2\varepsilon i_n^- + j_n^-.$$

Now, on account of (v) of Lemma 2.1, inequality (2.4.13) in turn is equivalent to $4A_{n+1}^2 A_n^2 > \left[A_{n+1}^2 + A_n^2 - (\varepsilon f_n + g_n^-)^2\right]^2$, that is,

$$\left(\varepsilon^2 h_n^- + 2\varepsilon i_n^- + j_n^-\right)^2 - 2\left(\varepsilon^2 h_n^+ + 2\varepsilon i_n^+ + j_n^+\right)\left(\varepsilon f_n + g_n^-\right)^2 + \left(\varepsilon f_n + g_n^-\right)^4 < 0. \tag{2.4.14}$$

Noting that $j_n^- = g_n^+ g_n^-$, we compute that $(j_n^-)^2 + (g_n^-)^4 - 2j_n^+(g_n^-)^2 = 0$, and we find that (2.4.14) is equivalent to

$$\begin{aligned}
&\varepsilon^3\left[(h_n^-)^2 + f_n^4 - 2f_n^2 h_n^+\right] + 4\varepsilon^2\left[h_n^- i_n^- + f_n^3 g_n^- - (h_n^+ f_n g_n^- + i_n^+ f_n^2)\right] \\
&+ 2\varepsilon\left[2(i_n^-)^2 + h_n^- j_n^- + 3f_n^2(g_n^-)^2 - (h_n^+(g_n^-)^2 + 4i_n^+ f_n g_n^- + j_n^+ f_n^2)\right] \\
&+ 4g_n^-\left[i_n^- g_n^+ + f_n(g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n)\right] < 0.
\end{aligned} \tag{2.4.15}$$

On account of (i), (iii) and (iv) of Lemma 2.1, the inequality (2.4.15) holds for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, for some $\tilde{\varepsilon}(n) > 0$. Finally, it results from (2.4.15), (ii) and (v) of Lemma 2.1 that (2.4.11) holds for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$. ∎

We now prove the following result.

Theorem 2.4 *Let* (1.5.3)-(1.5.6) *hold. We assume that* $g \in \mathcal{C}^3(\mathbb{R})$, $\Omega = \Pi_{i=1}^d (0, L_i)$, $d \leq 2$ *and* L_1/L_2 *is rational if* $d = 2$. *Then, there exists* $\tilde{\varepsilon}(n)$ *such that, for every* $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, *System* (2.4.4) *has an inertial manifold* $\widetilde{\mathfrak{M}}_\varepsilon^{\alpha,\sigma}$

(with dimension independent of ε but depends on δ) in $K_{\alpha,\sigma} \cap \mathcal{U}_d$.

Proof. We set

$$X_n = \text{span}\left\{U_k^{\pm}, \quad k = 0, 1, ..., n\right\}, \;\; Y_n = \overline{\text{span}\left\{U_k^{\pm}, \;\; k = n+1, \, n+2, ...\right\}}^{\mathcal{U}_1},$$

$$\sigma_1 = \{\mu_l^-, \mu_m^+, \;\; \max\{\mu_l^-, \mu_m^+\} \leq \mu_n^-\},$$

$$\sigma_2 = \{\mu_l^+, \mu_m^{\pm}, \;\; \mu_l^- \leq \mu_n^- < \min\{\mu_l^+, \mu_m^{\pm}\}\}$$

and $\quad X_{n1} = \text{span}\{U_l^-, U_m^+, \quad \mu_l^-, \mu_m^+ \in \sigma_1\}, \quad X_{n2} = \text{span}\{U_l^+, \quad \mu_l^- \leq \mu_n^- < \mu_l^+\}.$

We introduce the scalar product $\langle ., . \rangle$ in $\mathcal{U}_1$ (inspired by [76]) defined by

$$\langle U, V \rangle = \Psi_1(\mathrm{P}_{X_n} U, \mathrm{P}_{X_n} V) + \Psi_2(\mathrm{P}_{Y_n} U, \mathrm{P}_{Y_n} V), \tag{2.4.16}$$

for any $U, V \in \mathcal{U}_1$, where P_{X_n} and P_{Y_n} are, respectively, the projections from $\mathcal{U}_1$ onto X_n and Y_n and the functions $\Psi_1 : X_n \times X_n \to \mathbb{R}$ and $\Psi_2 : Y_n \times Y_n \to \mathbb{R}$ are defined by

$$\Psi_1(\mathscr{U}, \mathscr{V}) = (1+\tau)(u, y) + \delta(\nabla u, \nabla y) + \varepsilon(u, z) + \epsilon(y, v) + \varepsilon^2(v, z), \tag{2.4.17}$$

$$\Psi_2(\mathscr{U}, \mathscr{V}) = (1+\tau)(u, y) + \delta(\nabla u, \nabla y) + \varepsilon(u, z) + \varepsilon(y, v) + \varepsilon^2(v, z), \tag{2.4.18}$$

with $\mathscr{U} = (u, v)$, $\mathscr{V} = (y, z)$ in X_n (or in Y_n). Indeed, we have

$$\begin{aligned} \Psi_1(\mathscr{U}, \mathscr{U}) &= (1+\tau)\|u\|^2 + \delta\|\nabla u\|^2 + 2\varepsilon(u, v) + \varepsilon^2\|v\|^2 \\ &\geq (1+\tau)\|u\|^2 + \delta\|\nabla u\|^2 - 2\varepsilon\|u\|\|v\| + \varepsilon^2\|v\|^2 \\ &\geq (1+\tau)\|u\|^2 + \delta\|\nabla u\|^2 - \|u\|^2 - \frac{\varepsilon^2}{2}\|v\|^2 + \varepsilon^2\|v\|^2 \\ &= \tau\|u\|^2 + \delta\|\nabla u\|^2 + \frac{\varepsilon^2}{2}\|v\|^2, \quad \forall \mathscr{U} \in X_n, \end{aligned} \tag{2.4.19}$$

similarly

$$\Psi_2(\mathscr{U}, \mathscr{U}) \geq \tau\|u\|^2 + \delta\|\nabla u\|^2 + \frac{\varepsilon^2}{2}\|v\|^2, \quad \forall \mathscr{U} \in Y_n, \tag{2.4.20}$$

Thus, for $U_l^- \in X_{n1}$ and $U_l^+ \in X_{n2}$, noting that $(e_l, e_k) = \delta_{lk}$,

$$\tilde{\mu}_l^+ + \tilde{\mu}_l^- = \frac{1}{\varepsilon}((\delta - \varepsilon)\lambda_l + 1 + \tau), \quad \tilde{\mu}_l^+ \tilde{\mu}_l^- = -\frac{\lambda_l}{\varepsilon},$$

then we have that

$$\begin{aligned}
\langle U_l^-, U_l^+ \rangle &= \Psi_1\left(\left(\begin{smallmatrix} e_l \\ -\tilde{\mu}_l^- e_l \end{smallmatrix} \right), \left(\begin{smallmatrix} e_l \\ -\tilde{\mu}_l^+ e_l \end{smallmatrix} \right) \right) \\
&= (1+\tau)(e_l, e_l) + \delta(\nabla e_l, \nabla e_l) - \varepsilon(e_l, \tilde{\mu}_l^+ e_l) - \varepsilon(e_l, \tilde{\mu}_l^- e_l) + \varepsilon^2(\tilde{\mu}_l^+ e_l, \tilde{\mu}_l^- e_l) \\
&= (1+\tau) + \delta\lambda_l - \varepsilon(\tilde{\mu}_l^+ + \tilde{\mu}_l^+) + \varepsilon^2 \tilde{\mu}_l^- \tilde{\mu}_l^+ \\
&= 0.
\end{aligned}$$

As a consequence, X_{n1} is orthogonal to X_{n2} and to Y_n, and the decomposition $K_{\alpha,\sigma} = X_{n1} \oplus X_{n2} \oplus Y_n$ is orthogonal with respect to the scalar product $\langle ., . \rangle$ and we set $\mathcal{U}_1^1 = X_{n1}$ and $\mathcal{U}_1^{1\perp} = X_{n2} \oplus Y_n$. Let $\mathcal{P}$ be the unique orthogonal projection onto $\mathcal{U}_1^1$, then $\mathcal{Q} = I - \mathcal{P}$ is defined on $\mathcal{U}_1^{1\perp}$. We now define the norm

$$|||U||| = \langle U, U \rangle^{1/2}. \tag{2.4.21}$$

Remark 2.4.1 *From* (2.4.19) *and* (2.4.20), *we deduce that there exist* $c'_\delta, c_\delta > 0$ *independent of* ε *such that*

$$c'_\delta(\|\phi\|_1^2 + \varepsilon^2\|u\|^2) \le |||U||| \le c_\delta(\|\phi\|_1^2 + \varepsilon^2\|u\|^2), \quad \forall U = (\phi, u) \in \mathcal{U}_1,$$

where $c'_\delta = \min\{\frac{1}{2}, \delta, \tau\}$ *and* $c_\delta = \max\{2, \delta, (2+\tau)\}$

We have that Λ is an algebra on $H^1(\Omega)$ when $d = 1$ and on $H^2(\Omega)$ when $d = 2$, i.e.,

$$\|\Lambda\phi\varphi\| \le c\|\Lambda\phi\|\|\Lambda\varphi\|.$$

We have that $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'', \mathfrak{g}'''$ are bounded continuous functions on $K_\alpha \cap \mathscr{D}(\Lambda)$, and we deduce that

$$\|\Lambda\mathfrak{g}'(\phi)v\| \leq c_\delta\|\Lambda v\|, \quad \forall\phi, v \in K_\alpha \cap \mathscr{D}(\Lambda),$$

hence there exists $c > 0$ such that,

$$\|\Lambda\mathfrak{g}(\phi)\| \leq c_\delta, \qquad \forall\phi \in K_\alpha \cap \mathscr{D}(\Lambda) \tag{2.4.22}$$

$$\|\Lambda\mathfrak{g}(\phi) - \Lambda\mathfrak{g}(\varphi)\| \leq c_\delta\|\Lambda\phi - \Lambda\varphi\|, \quad \forall\phi, \varphi \in K_\alpha \cap \mathscr{D}(\Lambda), \tag{2.4.23}$$

(cf. [71]).

We define

$$\Gamma = \begin{cases} I, & \text{if} \quad d = 1, \\ \\ (I+N)^{1/2}, & \text{if} \quad d = 2. \end{cases} \tag{2.4.24}$$

Thus, on account of (2.4.22) and (2.4.23), we have that $\mathcal{G} : \mathcal{U}_d \to \mathcal{U}_d$ is globally Lipschitz continuous, that is, there exists $c > 0$, independent of ε, such that

$$\|\!|\Gamma\mathcal{G}(U)|\!\|\leq c_\delta, \qquad \forall U \in K_{\alpha,\sigma} \cap \mathcal{U}_d, \tag{2.4.25}$$

$$\|\!|\Gamma\mathcal{G}(U) - \Gamma\mathcal{G}(V)|\!\| \leq c_\delta \|\!|\Gamma U - \Gamma V|\!\|, \quad \forall U, V \in K_{\alpha,\sigma} \cap \mathcal{U}_d. \tag{2.4.26}$$

Indeed, for $U = (\phi, u), V = (\varphi, v)$, from (2.4.22) we have

$$\begin{aligned} \|\!|\Gamma G(U)|\!\| &= \|\Gamma(\tau I + \delta N)^{-1}N\mathfrak{g}(\phi)\|_1 + \varepsilon\|\Gamma\frac{1}{\varepsilon}(\tau I + \delta N)^{-1}N\mathfrak{g}(\phi)\| \\ &\leq c\delta^{-1}(\|\Gamma\mathfrak{g}(\phi)\|_1 + \|\Gamma\mathfrak{g}(\phi)\|) \\ &\leq c_\delta\|\Lambda\mathfrak{g}(\phi)\| \\ &\leq c_\delta, \end{aligned}$$

and from (2.4.23) we have

$$
\begin{aligned}
|||\Gamma G(U)-\Gamma G(V)||| &= \left|\left|\left|\Gamma(\tau I+\delta N)^{-1}N(\mathfrak{g}(\phi)-\mathfrak{g}(\varphi),\frac{1}{\varepsilon}[\mathfrak{g}(\phi)-\mathfrak{g}(\varphi)])\right|\right|\right| \\
&\le \frac{c}{\delta}(\|\Gamma\big(\mathfrak{g}(\phi)-\mathfrak{g}(\varphi)\big)\|_1+\|\Gamma\big(\mathfrak{g}(\phi)-\mathfrak{g}(\varphi)\big)\|) \\
&\le c_\delta\|\Lambda\phi-\Lambda\varphi\| \\
&\le c_\delta|||\Gamma U-\Gamma V|||.
\end{aligned}
$$

Moreover, there exist $C_1, C_2 > 0$, independent of ε, such that

$$
\|\mathcal{Q}e^{s\mathcal{A}\mathcal{Q}}\|_{\mathcal{L}(\mathcal{Q}\mathcal{U}_d)} \le C_1 e^{s\mu_{n+1}^-}, \quad s<0,
$$

$$
\|\mathcal{P}e^{-s\mathcal{A}\mathcal{P}}\|_{\mathcal{L}(\mathcal{P}\mathcal{U}_d)} \le C_2 e^{-s\mu_n^-}, \quad s\le 0,
$$

for every $\varepsilon \in (0,1]$.

Indeed, let $\phi = \sum_{j=0}^{n}\alpha_j w_j$ and $\varphi = \sum_{j=n+1}^{\infty}\alpha_j w_j$ be element of $\mathcal{P}K_{\alpha,\sigma}$ and $\mathcal{Q}K_{\alpha,\sigma}$ respectively.

$$
\|\mathcal{P}e^{-s\mathcal{A}\mathcal{P}}\phi\|^2 = \sum_{j=0}^{n}\left(e^{-s\mu_j^\pm}\right)^2\alpha_j^2 \le \sup_{\psi\le\mu_n^-}\left(e^{-s\psi}\right)^2\|\phi\|^2 \le \left(e^{-s\mu_n^-}\right)^2\|\phi\|^2, \quad s\le 0, \tag{2.4.27}
$$

and

$$
\|\mathcal{Q}e^{s\mathcal{A}\mathcal{Q}}\varphi\|^2 = \sum_{\mu_j^\pm\in\sigma_2}\left(e^{s\mu_j^\pm}\right)^2\alpha_j^2 \le \sup_{\psi\ge\mu_{n+1}^-}\left(e^{s\psi}\right)^2\|\varphi\|^2 \le \left(e^{s\mu_{n+1}^-}\right)^2\|\varphi\|^2, \quad s<0. \tag{2.4.28}
$$

It follows from the existence theorem of inertial manifolds Theorem 1.3 (see also [71, Chap. 9, Theorem 2.1] and [68]) that the semigroup generated by Equation (2.4.4), admits an inertial manifold $\widetilde{\mathfrak{M}}_\varepsilon^{\alpha,\sigma}$ in $K_{\alpha,\sigma}\cap\mathcal{U}_d$. More precisely, there exists a Lipschitz mapping $\Phi_\varepsilon^{\alpha,\sigma} : K_{\alpha,\sigma}\cap\mathcal{P}\mathcal{U}_d \to \mathcal{Q}\mathcal{U}_d$ such that the graph of $\Phi_\varepsilon^{\alpha,\sigma}$ defines an inertial manifold

$$\widetilde{\mathfrak{M}}_{\varepsilon}^{\alpha,\sigma} = \{\bar{\mathrm{p}} + \Phi_{\varepsilon}^{\alpha,\sigma}(\bar{\mathrm{p}}),\ \ \bar{\mathrm{p}} \in K_{\alpha,\sigma} \cap \mathcal{P}\mathcal{U}_d\}\,,$$

of dimension n independent of ε, for the semigroup $\tilde{S}_{\varepsilon}(t) : U_0 \mapsto U(t)$, where $U(t)$ is solution of (2.4.4) at time t, with respect to the metric induced by the norm $|||\Gamma\cdot|||$.

∎

2.5 Exponential attractors

2.5.1 Estimates of the difference of two solutions

Firstly, we estimate the difference of two solutions of (2.0.1).

Proposition 2.4 *There exist $c_\delta, c'_\delta > 0$ independent of ε such that*

$$\|S_{\varepsilon}(t)z_1 - S_{\varepsilon}(t)z_2\|^2_{\mathcal{U}_{2,\varepsilon}} \leq c_\delta(t^{-1}+1)e^{c'_\delta t}\|z_1 - z_2\|^2_{\mathcal{U}_{1,\varepsilon}},\ \ \forall t > 0, \tag{2.5.1}$$

for any $z_i = (\phi_{0i}, u_{0i}) \in \mathcal{B}_2$, $i = 1, 2$, and any $\varepsilon \in (0, 1]$.

Proof. We consider two solutions (ϕ_1, u_1) and (ϕ_2, u_2) of (2.0.1) with initial conditions

$$\phi_i|_{t=0} = \phi_{0i}, \qquad u_i|_{t=0} = u_{0i},$$

such that $(\phi_{0i}, u_{0i}) \in \mathcal{B}_2$, $i = 1, 2$. We set $\phi = \phi_1 - \phi_2$, $u = u_1 - u_2$, $\widetilde{\phi}_0 = \phi_{01} - \phi_{02}$ and $\widetilde{u}_0 = u_{01} - u_{02}$. There exists $c_\delta > 0$, independent of ε, such that

$$\|\phi_i(t)\|^2_2 + \varepsilon\|u_i(t)\|^2_1 \leq c_\delta,\ \ \forall t \geq 0. \tag{2.5.2}$$

The pair (ϕ, u) satisfies the problem

$$\tau\phi_t + N\Big(\delta\phi_t + N\phi + g(\phi_1) - g(\phi_2) - u\Big) = 0, \tag{2.5.3}$$

$$\varepsilon u_t + \phi_t + Nu = 0, \tag{2.5.4}$$

$$\phi|_{t=0} = \widetilde{\phi}_0, \quad u|_{t=0} = \widetilde{u}_0. \tag{2.5.5}$$

We multiply (2.5.3) and (2.5.4) by $N^{-1}\phi_t$ and by u, respectively, integrate over Ω, then sum the resulting equations and we deduce

$$\frac{d}{dt}\left(\varepsilon\|u\|^2 + \|\nabla\phi\|^2\right) + 2\|\nabla u\|^2 + 2\tau\|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2 \le \frac{c'}{\delta}\|\phi\|_1^2,$$

hence

$$\|(\phi(t), u(t))\|_{\mathcal{U}_{1,\varepsilon}}^2 + \int_0^t \left(\|u(s)\|_1^2 + \delta\|\phi_t(s)\|^2\right) ds \le c\|(\widetilde{\phi}_0, \widetilde{u}_0)\|_{\mathcal{U}_{1,\varepsilon}}^2 e^{\frac{c'}{\delta}t}, \ \ \forall t \ge 0, \tag{2.5.6}$$

where c and c' are independent of ε. We can also deduce from (2.5.3) and (2.5.6) that

$$\int_0^t \|\phi(s)\|_2^2 ds \le c_\delta\|(\widetilde{\phi}_0, \widetilde{u}_0)\|_{\mathcal{U}_{1,\varepsilon}}^2 e^{\frac{c'}{\delta}t}, \ \ \forall t \ge 0. \tag{2.5.7}$$

Now, we multiply (2.5.3) and (2.5.4) by ϕ_t and Nu, respectively, integrate over Ω, then sum the resulting equations, we find

$$\frac{d}{dt}\left(\varepsilon\|\nabla u\|^2 + \|N\phi\|^2\right) + c\left(\|Nu\|^2 + \|\phi_t\|_1^2\right) \le \frac{c}{\delta}\|\phi\|_1^2. \tag{2.5.8}$$

We multiply (2.5.8) by t and obtain

$$\frac{d}{dt}\left(\varepsilon t\|\nabla u\|^2 + t\|N\phi\|^2\right) + ct\left(\|Nu\|^2 + \|\phi_t\|_1^2\right) \le \frac{c}{\delta}(t+1)\left(\varepsilon\|u\|_1^2 + \|\phi\|_2^2\right). \tag{2.5.9}$$

Integrating (2.5.9) between 0 and t, and using (2.5.6) and (2.5.7), we deduce the result. ∎

Now, we prove the following

Proposition 2.5 *There exists $c > 0$, independent of ε, such that for every $\varepsilon \in (0, 1]$*

$$\|S_\varepsilon(t)\mathrm{z}\|_{\mathcal{U}_2} \leq \frac{c}{\delta}, \quad \forall t \geq 1, \quad \forall \mathrm{z} \in \mathcal{B}_2. \tag{2.5.10}$$

Proof. Multiplying $(2.0.1)_2$ by Nu and integrating over Ω, we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2 + \|Nu\|^2 + (\phi_t, Nu) = 0. \tag{2.5.11}$$

From $(2.0.1)_1$, we deduce that

$$\phi_t = -(\tau I + \delta N)^{-1}N(N\phi + g(\phi) - u). \tag{2.5.12}$$

Substituting (2.5.12) into (2.5.11), we find

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2 + \|Nu\|^2 + \|N(\tau I + \delta N)^{-1/2}u\|^2 \\ &= ((\tau I + \delta N)^{-1}N^2\phi, Nu) + ((\tau I + \delta N)^{-1}Ng(\phi), Nu).\end{aligned} \tag{2.5.13}$$

We have $\|g(\phi(t))\|_{L^\infty(\Omega)} \leq c$, since $\|\phi(t)\|_2 \leq c$, $\forall t \geq 0$, and on account of (1.5.7) we deduce from (2.5.13) that

$$\varepsilon\frac{d}{dt}\|\nabla u\|^2 + c\|\nabla u\|^2 \leq \frac{c}{\delta}. \tag{2.5.14}$$

We first multiply (2.5.14) by $e^{ct/\varepsilon}$, then we integrate between s and $t+1$, for any $s \leq t+1$. This yields

$$\varepsilon\|\nabla u(t+1)\|^2 e^{c(t+1)/\varepsilon} \leq c\varepsilon\|\nabla u(s)\|^2 e^{cs/\varepsilon} + \frac{c}{\delta}\varepsilon\left(e^{c(t+1)/\varepsilon} - e^{cs/\varepsilon}\right). \tag{2.5.15}$$

Integrating now (2.5.15) between t and $t+1$ with respect to s, we deduce

$$\|u(t)\|_1^2 \leq \frac{c}{\delta}, \quad \forall t \geq 1,$$

since $|m(u_0)| \leq \sigma$, hence the result. ∎

From now on, we will assume that

$$\mathcal{B} = \{\varphi \in \mathscr{D}(N^{k/2}),\ \|(I+N)^{k/2}\varphi\| \leq r_k\}$$

is an absorbing set of $S(t)$ on K_α, where $r_k > 0$ are the same as in $\mathcal{B}_k$, $k = 1{,}2$.

We now show the following estimate.

Proposition 2.6 *There exist $t_\star > 0$, $c_\delta > 0$ and $c'_\delta > 0$ (all independent of ε) such that*

$$\|S_\varepsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0)\|^2_{\mathcal{U}_{1,\varepsilon}} \leq c_\delta \sqrt[4]{\epsilon} e^{c'_\delta t}, \ \ \forall t \geq t_\star, \tag{2.5.16}$$

for any $\quad (\phi_0, u_0) \in \mathcal{B}_2, \quad$ and

$$\|S_\varepsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0)\|^2_{\mathcal{U}_1} \leq c_\delta \sqrt[4]{\epsilon} e^{c'_\delta t}, \ \ \forall t \geq t_\star, \tag{2.5.17}$$

for any $\quad (\phi_0, u_0) \in S_\varepsilon(1)\mathcal{B}_2, \quad$ and any $\quad \varepsilon \in (0, 1]$.

Proof. Let us take $(\phi_0, u_0) \in \mathcal{B}_2$. We set

$$(\phi^\varepsilon(t), u^\varepsilon(t)) = S_\varepsilon(t)(\phi_0, u_0), \qquad (\phi(t), u(t)) = (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0).$$

On account of (2.5.2) and (2.5.10), there exist $c_\delta > 0$ such that

$$\|\phi^\varepsilon(t)\|_2^2 + \|u^\varepsilon(t)\|_1^2 \leq c_\delta, \quad \forall t \geq 0, \tag{2.5.18}$$

$$\|\phi(t)\|_2 \leq c_\delta, \quad \forall t \geq 0. \tag{2.5.19}$$

We now set $P = \phi^\varepsilon - \phi$ and $R = u^\varepsilon - u$ and they satisfy the following problem:

$$\tau P_t + N(\delta P_t + NP + g(\phi^\varepsilon) - g(\phi) - R) = 0, \tag{2.5.20}$$

$$\varepsilon R_t + P_t + NR = -\varepsilon u_t, \tag{2.5.21}$$

$$P|_{t=0} = 0, \quad R|_{t=0} = u_0 - \mathrm{L}_{m(u_0)}\phi_0. \tag{2.5.22}$$

We also have

$$m(P(t)) = m(R(t)) = m(u_t(t)) = 0, \quad \forall t \geq 0.$$

We multiply (2.5.20) and (2.5.21) by $N^{-1}P_t$ and R, we integrate over Ω, and we obtain

$$\frac{1}{2}\frac{d}{dt}\|P\|_1^2 + \tau\|P_t\|_{-1}^2 + \delta\|P_t\|^2 - (R, P_t) + (g(\phi^\varepsilon) - g(\phi), P_t) = 0, \tag{2.5.23}$$

and

$$\frac{\varepsilon}{2}\frac{d}{dt}\|R\|^2 + \|R\|_1^2 + (P_t, R) = -\varepsilon(u_t, R), \tag{2.5.24}$$

respectively. Summing (2.5.23) and (2.5.24) and noting that $\|g(\phi^\varepsilon) - g(\phi)\| \leq c\|P\|_1$, we deduce that

$$\frac{d}{dt}\left(\|P\|_1^2 + \varepsilon\|R\|^2\right) + 2\|P_t\|_{-1}^2 + \delta\|P_t\|^2 + \|R\|_1^2 \leq c_1\delta^{-1}\|P\|_1^2 + \varepsilon^2\|u_t\|_{-1}^2. \tag{2.5.25}$$

We will show that

$$\|u_t(t)\|^2 \leq c_\delta e^{ct}, \ \ \forall t \geq 0. \tag{2.5.26}$$

We then apply the Gronwall's lemma to (2.5.25), and we find

$$\|P(t)\|_1^2 + \varepsilon\|R(t)\|^2 \le c_\delta \left(\varepsilon\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) e^{c_\delta t}, \ \ \forall t \ge 0, \tag{2.5.27}$$

due to (2.5.26). Integrating now (2.5.25) between 0 and t, we obtain

$$\int_0^t \left(\delta\|P_t\|^2 + \|R\|_1^2\right) ds \le c_\delta \left(\varepsilon\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) e^{c_\delta t}, \ \ \forall t \ge 0, \tag{2.5.28}$$

due to (2.5.26) and (2.5.27). From (2.5.20), we deduce that

$$P_t = -(\tau I + \delta N)^{-1} N \left(NP + g(\phi^\varepsilon) - g(\phi) - R\right). \tag{2.5.29}$$

Substituting (2.5.29) into (2.5.24), we find

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\|R\|^2 + \|R\|_1^2 + \|\nabla(\tau I + \delta N)^{-1/2} R\|^2 \\ &= (\nabla(\tau I + \delta N)^{-1} NP, \nabla R) + (\nabla(\tau I + \delta N)^{-1}[g(\phi^\varepsilon) - g(\phi)], \nabla R) \\ &\quad - \varepsilon(N^{-1/2}u_t, N^{1/2}R). \end{aligned} \tag{2.5.30}$$

On account of (1.5.7) and (1.5.9), we deduce from (2.5.30) that

$$\varepsilon\frac{d}{dt}\|R\|^2 + \|R\|_1^2 + C_1\|R\|^2 \le \frac{c}{\delta}\|P\|_1^2 + c\varepsilon^2\|u_t\|_{-1}^2.$$

Also, due to (2.5.26) and (2.5.27), we have

$$\frac{d}{dt}\left(\varepsilon\|R\|^2\right) + \frac{c}{\varepsilon}\left(\varepsilon\|R\|^2\right) \le c_\delta \left(\varepsilon\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) e^{c_\delta t}. \tag{2.5.31}$$

We multiply (2.5.31) by t and we deduce that

$$\frac{d}{dt}\left(\varepsilon t\|R\|^2 e^{ct/\varepsilon}\right) \le \varepsilon\|R\|^2 e^{ct/\varepsilon} + c_\delta t e^{ct/\varepsilon}\left(\varepsilon\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) e^{c_\delta t}. \tag{2.5.32}$$

Integrating (2.5.32) between 0 and t, we find

$$\varepsilon t\|R(t)\|^2 \leq \varepsilon \int_0^t \|R(s)\|^2 ds + c_\delta \varepsilon t \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) e^{c'_\delta t},$$

so that, due to (2.5.28),

$$\varepsilon\|R(t)\|^2 \leq \varepsilon c_\delta \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right) t^{-1} e^{c'_\delta t}, \quad \forall t > 0,$$

hence

$$\varepsilon\|R(\sqrt{\varepsilon})\|^2 \leq c_\delta \sqrt{\varepsilon} \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|^2 + \varepsilon\right). \tag{2.5.33}$$

Like Estimate (2.5.27), we can show that

$$\|P(t)\|_2^2 + \varepsilon\|R(t)\|_1^2 \leq c_\delta \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|_1^2 + \varepsilon\right) e^{c_\delta t}, \ \ \forall t \geq 0.$$

On the other hand, we have (cf. again (2.5.28))

$$\begin{aligned}
\|P(t)\|_1^2 &\leq c\|P(t)\|\|P(t)\|_2 \\
&\leq c\|P(t)\|_2 \int_0^t \|P_t(s)\| ds \\
&\leq c_\delta \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|_1^2 + \varepsilon\right) \sqrt{t} e^{c'_\delta t}, \quad \forall t \geq 0,
\end{aligned}$$

so that

$$\|P(\sqrt{\varepsilon})\|_1^2 \leq c_\delta \sqrt[4]{\varepsilon} \left(\varepsilon \|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|_1^2 + \varepsilon\right). \tag{2.5.34}$$

We now apply the Gronwall's lemma to (2.5.25) between $\sqrt{\varepsilon}$ and $t+\sqrt{\varepsilon}$. We find

$$\left(\|P\|_1^2 + \varepsilon\|R\|^2\right)(t+\sqrt{\varepsilon}) \leq c_\delta \left[\left(\|P\|_1^2 + \varepsilon\|R\|^2\right)(\sqrt{\varepsilon}) + \varepsilon^2\right] e^{c'_\delta t}, \quad \forall t \geq 0. \tag{2.5.35}$$

Thanks to (2.5.33) and (2.5.34), from (2.5.35) it follows that

$$\left(\|P\|_1^2+\varepsilon\|R\|^2\right)(t+\sqrt{\varepsilon}) \leq c_\delta \sqrt[4]{\varepsilon}\left(\varepsilon\|u_0-\mathrm{L}_{m(u_0)}\phi_0\|_1^2+\varepsilon\right)e^{c'_\delta t}, \quad \forall t \geq 0. \tag{2.5.36}$$

Again apply the Gronwall's lemma to (2.5.25) between s and t, we obtain the following estimate

$$\|P(t)\|_1^2+\varepsilon\|R(t)\|^2 \leq c_\delta\left(\|P(s)\|_1^2+\varepsilon\|R(s)\|^2+\varepsilon^2\right)e^{c'_\delta t},$$

for any given $s \geq 0$ and any $t > s$. Let $t_\star > 0$, independent of ε, be such that $t_\star > \sqrt{\varepsilon}$. This latter estimate, with $s=\sqrt{\varepsilon}$, combined with (2.5.36) gives

$$\|P(t)\|_1^2+\varepsilon\|R(t)\|^2 \leq c_\delta \sqrt[4]{\varepsilon}\left(\varepsilon\|u_0-\mathrm{L}_{m(u_0)}\phi_0\|_1^2+\varepsilon\right)e^{c'_\delta t}, \quad \forall t > \sqrt{\varepsilon}. \tag{2.5.37}$$

Finally, estimate (2.5.16) follows from (2.5.18), while estimate (2.5.17) follows from (2.5.18) and (2.5.37) yield the result. ∎

Proof of **(2.5.26)** Firstly, we observe that $v=\phi_t$ is solution to the problem

$$(1+\tau)v_t+N(\delta v_t+Nv+g'(\phi)v)=0, \tag{2.5.38}$$

$$v|_{t=0}=\mathrm{I}\phi_0, \tag{2.5.39}$$

where $\phi(t)=S(t)\phi_0$,

$$\mathrm{I}\phi_0=-[(1+\tau)I+\delta N]^{-1}N(N\phi_0+g(\phi_0)),$$

and, $w=u_t$ satisfies

$$w(t)=[(1+\tau)I+\delta N]^{-1}(Nv(t)+g'(\phi(t))v(t)), \quad \forall t \geq 0.$$

We have $\|g'(\phi(t))\|_{L^\infty(\Omega)} \leq c,\ \forall t \geq 0$, due to (2.5.19), therefore $\|g'(\phi(t))v(t)\| \leq c\|v(t)\|,\ \forall t \geq 0$.

We multiply (2.5.38) by $N^{-1}v$ and integrate over Ω, and we deduce

$$\frac{d}{dt}\left[(1+\tau)\|v\|_{-1}^2+\delta\|v\|^2\right]+\|\nabla v\|^2\leq c\|v\|^2.$$

Application of the Gronwall's lemma gives $\|v(t)\|^2 \leq ce^{c't}$, $\forall t \geq 0$. Hence the result. □

2.5.2 A robust family of exponential attractors

Now, we prove the following result.

Theorem 2.5 *For every $\varepsilon \in (0,1]$, the semigroup $S_\varepsilon(t)$ possesses an exponential attractor $\mathcal{E}_\varepsilon^{\alpha,\sigma}$ in $K_{\alpha,\sigma}$ (with dimensions independent of ε), and there exist $0 < \varpi_1 \leq \frac{1}{2}$ and $M_\delta > 0$ (all independent of ε but depends on δ) such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{U}_{1,\epsilon}}(\mathcal{E}_\varepsilon^{\alpha,\sigma},(\mathcal{E}^\alpha)^\sigma)\leq M_\delta\varepsilon^{\varpi_1}, \tag{2.5.40}$$

$$\mathrm{dist}_{\mathcal{U}_1}(\mathcal{E}_\varepsilon^{\alpha,\sigma},(\mathcal{E}^\alpha)^\sigma)\leq M_\delta\varepsilon^{\varpi_1}, \quad \textit{for any } \varepsilon\in(0,1], \tag{2.5.41}$$

$$\text{and} \quad \lim_{\varepsilon\to 0}\mathrm{dist}_{\mathcal{U}_1}((\mathcal{E}^\alpha)^\sigma,\mathcal{E}_\varepsilon^{\alpha,\sigma})=0, \tag{2.5.42}$$

where $\mathcal{E}^\alpha$ is an exponential attractor for the semigroup $S(t)$ on K_α.

Proof. We first observe that the semigroup $S(t)$ has an exponential attractor $\mathcal{E}^\alpha$ on K_α (see [7]). From now on, we set

$$\widetilde{\mathcal{B}}_2 = S_\varepsilon(t^\star)\mathcal{B}_2,$$

where $t^\star > 0$ is independent of ε and such that $S_\varepsilon(t)\mathcal{B}_2 \subset \mathcal{B}_2$ for all $t \geq t^\star$. We will assume that $t^\star \geq 2$. Then, it follows that

$$\widetilde{\mathcal{B}}_2 \subset \{(\varphi,\psi) \in K_{\alpha,\sigma} \cap \mathcal{U}_2, \|(\varphi,\psi)\|_{\mathcal{U}_{2,\varepsilon}} \leq r_{2,\delta}\}.$$

Note that $\widetilde{\mathcal{B}}_2$ is also a bounded absorbing set for $S_\varepsilon(t)|_{K_{\alpha,\sigma}}$.

Let us now prove the theorem. On account of Theorem 1.2, we let $E_\varepsilon = \mathcal{U}_1$, $V_\varepsilon = W_\varepsilon = \mathcal{U}_2$, $B_\varepsilon = \widetilde{\mathcal{B}}_2$ and we check all the assumptions 1-5. To verify Assumption 1, there exists a constant c such that

$$\begin{aligned}
\|\mathrm{L}_\beta\phi_1 - \mathrm{L}_\beta\phi_2\| &= \|[(1+\tau)I + \delta N]^{-1}(N(\phi_1-\phi_2) + \overline{g(\phi_1)} - \overline{g(\phi_2)})\| \\
&\leq \frac{1}{\delta}\|\phi_1-\phi_2\| + \frac{1}{\delta(1+\tau)}\big(\|g(\phi_1)-g(\phi_2)\| + \|m\big(g(\phi_1)-g(\phi_2)\big)\|\big) \\
&\leq c_\delta(\|\phi_1-\phi_2\| + \|g(\phi_1)-g(\phi_2)\|) \\
&\leq c_\delta\|\phi_1-\phi_2\|_1, \qquad (2.5.43)
\end{aligned}$$

for any ϕ_1 and ϕ_2 in $\mathcal{B}$, $\theta \in [0,1]$. Hence taking $L = L_\beta$ and the Hölder exponent $\alpha = 1$, we obtain Assumption 1.

Assumption 2 is satisfied by Propositions 2.4 and 2.5.

Assumption 3, we choose t^* such that (2.5.17) is satisfied. Then, from Estimate (2.5.17) and the fact that $\widetilde{\mathcal{B}}_2 = S_\varepsilon(t^\star - 1)S_\varepsilon(1)\mathcal{B}_2$, we obtain that

$$\begin{aligned}
&\|S_\varepsilon^m(t^*)(\phi_0,u_0) - (S^m(t^*)\phi_0, \mathrm{L}_{m(u_0)}S^m(t^*)\phi_0)\|^2_{\mathcal{U}_1} \leq c^m(\delta)\sqrt[4]{\varepsilon},\ \ \forall m \in \mathbb{N}, \\
&\|S_\varepsilon(t)(\phi_0,u_0) - (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0)\|^2_{\mathcal{U}_1} \leq c(\delta)\sqrt[4]{\varepsilon},\ \ \forall t \in [t^*, 2t^*],
\end{aligned}$$

for any $(\phi_0,u_0) \in \widetilde{\mathcal{B}}_2$ and any $\varepsilon \in (0,1]$. Hence, Assumption 3 is verified.

Thus, we are left to check Assumptions 4 and 5. Indeed, defining

$$(\phi_i(t), u_i(t)) = S_\varepsilon(t) z_{0i},$$

with $z_{0i} = (\phi_{0i}, u_{0i}) \in \widetilde{\mathcal{B}}_2$, $i = 1, 2$, and $t \in [t^*, 2t^*]$, we obtain

$$\begin{aligned} &\|S_\varepsilon(t)z_{01} - S_\varepsilon(t')z_{02}\|_{\mathcal{U}_{1,\varepsilon}} \\ &\leq \|S_\varepsilon(t)z_{01} - S_\varepsilon(t')z_{01}\|_{\mathcal{U}_{1,\varepsilon}} + \|S_\varepsilon(t')z_{01} - S_\varepsilon(t')z_{02}\|_{\mathcal{U}_{1,\varepsilon}}, \ \forall t, t' \in [t^*, 2t^*]. \end{aligned} \tag{2.5.44}$$

On the one hand, we have

$$\int_0^t \|\phi_t(s)\|_1^2 ds + \int_0^t \varepsilon \|u_t(s)\|^2 ds \leq c(t+1), \ \forall t \geq 0. \tag{2.5.45}$$

Therefore,

$$\begin{aligned} \|S_\varepsilon(t)z_{01} - S_\varepsilon(t')z_{01}\|_{\mathcal{U}_{1,\varepsilon}} &\leq c\left(\|\phi(t) - \phi(t')\|_1 + \sqrt{\varepsilon}\|u(t) - u(t')\|\right) \\ &\leq c\int_t^{t'} \|\phi_t(s)\|_1 ds + c\int_t^{t'} \sqrt{\varepsilon}\|u_t(s)\| ds \\ &\leq c(t^*, \varepsilon, \delta)|t' - t|^{1/2}. \end{aligned}$$

On the other hand, it follows from (2.5.1) that

$$\|S_\varepsilon(t')z_{01} - S_\varepsilon(t')z_{02}\|_{\mathcal{U}_{1,\varepsilon}} \leq c(\delta, t^*)\|z_{01} - z_{02}\|_{\mathcal{U}_{1,\varepsilon}}, \ \ \forall t' > 0.$$

Hence, we conclude with

$$\|S_\varepsilon(t)z_{01} - S_\varepsilon(t')z_{02}\|_{\mathcal{U}_{1,\varepsilon}} \leq c(t^*, \varepsilon, \delta)(|t' - t|^{1/2} + \|z_{01} - z_{02}\|_{\mathcal{U}_{1,\varepsilon}}). \tag{2.5.46}$$

This shows the existence of exponential attractors on $\overline{\widetilde{\mathcal{B}}_2}^{\mathcal{U}_1}$ that satisfy (2.5.41) and (2.5.42). Then, like in [40], we can extend the basin of attraction to the whole phase-space $\mathcal{U}_1$ by using the transitivity property of the exponential attraction. ∎

Proof of **(2.5.45)** We deduce from the second equation of (2.0.1) that

$$\varepsilon^2 \int_0^t \|u_t\|^2 ds \leq c \int_0^t (\|\phi_t\|^2 + \|u\|_2^2) ds, \ \forall t \geq 0.$$

Integrating (2.1.23) between 0 and t, we get, owing to (2.5.18),

$$\int_0^t (\|\phi_t\|^2 + \delta\|\phi_t\|_1^2 + \|u\|_2^2) ds \leq c(t+1), \ \forall t \geq 0.$$

It follows from the two previous inequalities that

$$\int_0^t \|\phi_t(s)\|_1^2 ds + \int_0^t \varepsilon\|u_t(s)\|^2 ds \leq \frac{c(\delta+\varepsilon)}{\varepsilon\delta}(t+1), \ \forall t \geq 0. \qquad \square$$

2.6 Continuity of inertial manifolds

We now prove continuity properties for the inertial manifolds $\widetilde{\mathfrak{M}}_{\varepsilon}^{\alpha,\sigma}$. Firstly, we recall that the semigroup $S(t)$ possesses an inertial manifold $\mathfrak{M}_\alpha$ on K_α (see [35, 63]). More precisely, there exists a Lipschitz mapping $\Phi^\alpha : PK_\alpha \cap \mathscr{D}(\Lambda) \to Q\mathscr{D}(\Lambda)$ such that the graph of Φ^α defines an inertial manifold

$$\mathfrak{M}^\alpha = \{p + \Phi^\alpha(p), \ \ p \in PK_\alpha \cap \mathscr{D}(\Lambda)\},$$

for the unperturbed "prepared problem":

$$(1+\tau)\phi_t + N(\delta\phi_t + N\phi + \mathfrak{g}(\phi)) = 0,$$

where $\mathfrak{g}$ is defined by (2.4.2). Here P is the unique orthogonal projection in $\mathscr{D}(\Lambda)$ onto the space spanned by $\{e_0(x), e_1(x), ..., e_n(x)\}$ (cf. Sect. 1.5), and $Q = I - P$.

For any arbitrary $R > 0$, we let

$$\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,R} = \{(\phi, u),\ (\phi, u) = \bar{\mathrm{p}} + \Phi^{\alpha,\sigma}_{\varepsilon}(\bar{\mathrm{p}}),\ \ |||\bar{\mathrm{p}}||| \leq R\},$$

$$\mathfrak{M}^{\alpha}_{R} = \{\phi,\ \ \phi = p + \Phi^{\alpha}(p),\ \ \|p\| \leq R\},$$

$$\mathfrak{M}^{\alpha,\mu}_{R} = \{\varphi \in \mathfrak{M}^{\alpha}_{R}, \quad m(\varphi) = \mu\},$$

and $$(\mathfrak{M}^{\alpha,\mu}_{R})^{\sigma,\beta} = \{(\varphi, \mathrm{L}_\beta\varphi),\ \ \varphi \in \mathfrak{M}^{\alpha,\mu}_{R}\}.$$

We prove the following result.

Theorem 2.6 *Let the assumptions of Theorem 2.4 hold. Let $\mu \in [-\alpha, \alpha]$ and $\beta \in [-\sigma, \sigma]$. Then, there exists $M_\delta = M_\delta(n) > 0$ independent of ε, such that*

$$\mathrm{dist}_{\mathcal{U}_{d,\varepsilon}}((\mathfrak{M}^{\alpha,\mu}_{R})^{\sigma,\beta}, \widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon}) \leq M_\delta\sqrt{\varepsilon}, \quad \forall\varepsilon(0, \tilde{\varepsilon}(n)]. \tag{2.6.1}$$

Proof. We were inspired, in part, by the proof of Theorem 8.6 in [22].

Now let us prove (2.6.1). We consider an element $U_0 = (\phi_0, \mathrm{L}_\beta\phi_0)$ of $(\mathfrak{M}^{\alpha,\mu}_{R})^{\sigma,\beta}$. Since $\phi_0 \in \mathfrak{M}^{\alpha,\mu}_{R}$, there exists a complete trajectory $(\phi(t))_{t\in\mathbb{R}}$ lying in $\mathfrak{M}^{\alpha}$ such that for all time $t \in \mathbb{R}$, the function $U(t) = (\phi(t), u(t))$ satisfies the non autonomous initial value problem:

$$\begin{cases} U_t + \mathcal{A}U + \mathcal{G}(U) = F(t), \\ U(0) = (\phi_0, \mathrm{L}_\beta\phi_0), \end{cases} \tag{2.6.2}$$

where

$$F(t) = (0, u_t(t)).$$

We will prove that, there exists $M_\delta = M_\delta(n) > 0$, independent of ε, such that

$$\|\Gamma u_t(s)\| \leq M_\delta e^{-\gamma_n s}, \ \ \forall s \leq 0, \tag{2.6.3}$$

where
$$\gamma_n = \frac{\lambda_n^2}{2(1+\tau+\delta\lambda_n)} + \frac{\lambda_{n+1}^2}{2(1+\tau+\delta\lambda_{n+1})}.$$

Also, for any $c_1 > 0$ and for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, there holds

$$\begin{cases} \mu_n^- - \gamma_n + c_1 < 0, \\ \mu_{n+1}^- - \gamma_n - c_1 > 0. \end{cases} \tag{2.6.4}$$

Let $(U_\varepsilon(t))_{t\in\mathbb{R}}$ be a complete trajectory lying in $\widetilde{\mathfrak{M}}_\varepsilon^{\alpha,\sigma}$ and solution to (2.4.4). The function $\mathrm{z}(t) = U(t) - U_\varepsilon(t)$ is defined for all time $t \in \mathbb{R}$ and satisfies the problem

$$\begin{cases} \mathrm{z}_t + \mathcal{A}\mathrm{z} + \mathcal{G}(U) - \mathcal{G}(U_\varepsilon) = F(t), \\ \mathrm{z}(0) = U_0 - U_\varepsilon(0). \end{cases} \tag{2.6.5}$$

We write $\mathrm{z}(t) = \mathrm{p}(t) + \mathrm{q}(t)$, where $\mathrm{p} = \mathcal{P}\mathrm{z}$ and $\mathrm{q} = \mathcal{Q}\mathrm{z}$; and we have that

$$\begin{aligned} \mathrm{z}(t) =& e^{-\mathcal{A}\mathcal{P}t}\mathrm{p}(0) + \int_0^t e^{-\mathcal{A}\mathcal{P}(t-s)}\mathcal{P}\left[\mathcal{G}(U_\varepsilon(s)) - \mathcal{G}(U(s)) + F(s)\right] ds \\ &+ \int_{-\infty}^t e^{-\mathcal{A}\mathcal{Q}(t-s)}\mathcal{Q}\left[\mathcal{G}(U_\varepsilon(s)) - \mathcal{G}(U(s)) + F(s)\right] ds, \end{aligned} \tag{2.6.6}$$

(cf., e.g., [22, 23, 71]). Since $\mathcal{P}\mathcal{U}_d$ is a finite-dimensional subspace of $\mathcal{U}_d$, we can choose $U_\varepsilon(0)$ such that $\mathrm{p}(0) = 0$. On account of (2.6.16), and the fact that $\|\Lambda\Phi^\alpha(p(t))\| \leq c$, $\forall t \in \mathbb{R}$, we can deduce that $\|\Lambda\phi(t)\| \leq c_\delta e^{-\gamma_n t}$, $\forall t \leq 0$, and then $|||\Gamma U(t)||| \leq c_\delta e^{-\gamma_n t}$, $\forall t \leq 0$. Similarly, we can show that $|||\Gamma U_\epsilon(t)||| \leq c_\delta e^{-\mu_n^- t}$, $\forall t \leq 0$. Hence, $\mathrm{z} \in \mathcal{C}_{\gamma_n}((-\infty, 0]; \mathcal{U}_d)$, due to (2.6.4). Thus, we deduce from (2.6.6) and (2.4.26) that

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&= \sup_{t\le 0} e^{\gamma_n t}\Big\|\Big\{\int_0^t e^{-\mathcal{A}\mathcal{P}(t-s)}\mathcal{P}[\Gamma\mathcal{G}(U_\varepsilon(s))-\Gamma\mathcal{G}(U(s))+\Gamma F(s)]ds \\
&\qquad\qquad + \int_{-\infty}^t e^{-\mathcal{A}\mathcal{Q}(t-s)}[\Gamma\mathcal{G}(U_\varepsilon(s))-\Gamma\mathcal{G}(U(s))+\Gamma F(s)]ds\Big\}\Big\| \\
&\le c_\delta \sup_{t\le 0}\Big\{\int_0^t e^{(\mu_n^- -\gamma_n)(s-t)}ds + \int_{-\infty}^t e^{(\mu_{n+1}^- -\gamma_n)(s-t)}ds\Big\}\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&\quad + c_\delta'\sqrt{\varepsilon}\sup_{t\le 0}\Big\{e^{(\gamma_n-\mu_n^-)t}\int_t^0 e^{(s-t)\mu_n^-}\|\Gamma u_t(s)\|ds \\
&\qquad\qquad + e^{(\gamma_{n+1}-\mu_n^-)t}\int_{-\infty}^t e^{(s-t)\mu_{n+1}^-}\|\Gamma u_t(s)\|ds\Big\}.
\end{aligned} \tag{2.6.7}
$$

We infer from (2.6.7), (2.6.3) and (2.6.4) that

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
\le& c_\delta \sup_{t\le 0}\Big\{\frac{1}{\gamma_n-\mu_n^-}e^{-(\mu_n^- -\gamma_n)t}[e^{(\mu_n^- -\gamma_n)t}-1]+\frac{1}{\mu_{n+1}^- -\gamma_n}\Big\}\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&+ c_\delta'\sqrt{\varepsilon}\sup_{t\le 0}\Big\{\frac{1}{\gamma_n-\mu_n^-}e^{(\gamma_n-\mu_n^-)t}[e^{(\mu_n^- -\gamma_n)t}-1]+\frac{1}{\mu_{n+1}^- -\gamma_n}\Big\},
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|\mathrm{z}\|_{\mathcal{C}_\eta((-\infty,0];\mathcal{U}_d)} \le c_\delta\left(\frac{1}{\gamma_n-\mu_n^-}+\frac{1}{\mu_{n+1}^- -\gamma_n}\right)\|\mathrm{z}\|_{\mathcal{C}_\gamma((-\infty,0];\mathcal{U}_d)} \\
+ c_\delta'\sqrt{\varepsilon}\left(\frac{1}{\gamma_n-\mu_n^-}+\frac{1}{\mu_{n+1}^- -\gamma_n}\right).
\end{aligned} \tag{2.6.8}
$$

We then deduce from (2.6.8) due to (2.6.4), that

$$
\|\mathrm{z}\|_{\mathcal{C}_\eta((-\infty,0];\mathcal{U}_d)} \le \frac{1}{2}\|\mathrm{z}\|_{\mathcal{C}_\eta((-\infty,0];\mathcal{U}_d)} + M_\delta\sqrt{\varepsilon}, \tag{2.6.9}
$$

Observing that $|||\Gamma \mathrm{z}(0)||| \leq \|\mathrm{z}\|_{\mathcal{C}_\eta((-\infty,0];\mathcal{U}_d)}$, we deduce from (2.6.9) that $|||\Gamma \mathrm{z}(0)||| \leq M_\delta\sqrt{\varepsilon}$, that is,

$$|||\Gamma(\phi_0, \mathrm{L}_\beta\phi_0) - \Gamma U_\varepsilon(0)||| \leq M_\delta\sqrt{\varepsilon}. \tag{2.6.10}$$

Estimates (2.6.10) and Remark 2.4.1 imply the lower semicontinuity estimate (2.6.1). ∎

Proof of **(2.6.3)** Because $\phi_0 \in \mathfrak{M}_R^{\alpha,\mu}$, the function $\phi(t)$ is in the form

$$\phi(t) = p(t) + \Phi^\alpha(p(t)) \text{ and } m(\phi(t)) = \mu, \ \forall t \in \mathbb{R},$$

where $p(t)$ satisfies $p \in \mathcal{C}(\mathbb{R}, PK_\alpha \cap \mathscr{D}(\Lambda))$ and is the unique solution to the problem

$$p_t + Ap + PH(p + \Phi^\alpha(p))) = 0, \tag{2.6.11}$$

$$p|_{t=0} = p_0, \tag{2.6.12}$$

with $\|p_0\| \leq R$. Here $H(\phi) = B^{-1}N\mathfrak{g}(\phi)$,

$$B = (1+\tau)I + \delta N \text{ and } A = B^{-1}N^2 : \mathscr{D}(N) \to L^2(\Omega).$$

Since $g \in \mathcal{C}^2$, it is known that $\Phi^\alpha \in \mathcal{C}^2$ and

$$\|\Lambda(\Phi^\alpha)'(p)\| \leq 1, \quad \forall p \in PK_\alpha \cap \mathscr{D}(\Lambda),$$

(cf. [23]). We have

$$\phi_t(t) = p_t(t) + (\Phi^\alpha)'(p(t))p_t(t)$$

and

$$\|\Lambda\phi_t(t)\| \leq \|\Lambda p_t(t)\| + \|\Lambda((\Phi^\alpha)'(p(t))p_t(t))\| \leq c_\delta\|\Lambda p_t(t)\|. \tag{2.6.13}$$

From (2.6.11), we can also deduce that

$$\|\Lambda p_t(t)\| \leq \|A\Lambda p(t)\| + \|\Lambda H(p + \Phi^\alpha(p))\| \leq \gamma_n\|\Lambda p(t)\| + c_\delta. \tag{2.6.14}$$

We take the L^2-scalar product of (2.6.11) with $\Lambda^2 p$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda p\|^2 + (A\Lambda p, \Lambda p) + (P\Lambda H(p + \Phi^\alpha(p)), \Lambda p) = 0,$$

hence

$$\left|\frac{1}{2}\frac{d}{dt}\|\Lambda p\|^2 + (A\Lambda p, \Lambda p)\right| \leq c_\delta\|\Lambda p\|.$$

We then deduce that

$$-\|\Lambda p\|\frac{d}{dt}\|\Lambda p\| \leq \gamma_n\|\Lambda p\|^2 + c_\delta\|\Lambda p\|$$

and, therefore,

$$-\frac{d}{dt}\|\Lambda p\| \leq \gamma_n\|\Lambda p\| + c_\delta. \tag{2.6.15}$$

We now apply the Gronwall lemma to (2.6.15) between t and 0, $t \leq 0$, and we find

$$\|\Lambda p(t)\| \leq c_\delta\|\Lambda p_0\|e^{-\gamma_n t}, \quad \forall t \leq 0. \tag{2.6.16}$$

Finally, we remind that

$$u_t(t) = B^{-1}(N\phi_t(t) + \mathfrak{g}'(\phi(t))\phi_t(t)), \quad \forall t \in \mathbb{R},$$

and we can deduce that

$$\|\Lambda u_t(t)\| \leq c_\delta\|\Lambda\phi_t(t)\| + \|\mathfrak{g}'(\phi(t))\phi_t(t))\| \leq c_\delta\|\Lambda\phi_t(t)\| \quad \forall t \in \mathbb{R}. \tag{2.6.17}$$

Estimate (2.6.3) follows from (2.6.13), (2.6.14), (2.6.16) and (2.6.17). □

Proof of $\mathbf{(2.6.4)_1}$ We have

$$\mu^-_{n+1} - \gamma_n - c_1 = \frac{S_n + \varepsilon T_n - (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_{n+1}\sqrt{D_{n+1}}}{2\varepsilon(\tau+\delta\lambda_{n+1})(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})},$$

where

$$S_n = (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})^2\lambda_{n+1},$$

$$T_n = (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_{n+1}^2 - (\tau+\delta\lambda_{n+1})\Big[\lambda_{n+1}^2(1+\tau+\delta\lambda_n)$$
$$+ \lambda_n^2(1+\tau+\delta\lambda_{n+1}) + 2c_1(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\Big].$$

When $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, we can see that the sign of

$S_n + \varepsilon T_n - (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_{n+1}\sqrt{D_{n+1}}$

is the same as that of

$(S_n+\varepsilon T_n)^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_{n+1}^2 D_{n+1}$, and

$$(S_n+\varepsilon T_n)^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_{n+1}^2 D_{n+1}$$
$$= 2\varepsilon(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})^2\lambda_{n+1}\big[T_n - (1+\tau+\delta\lambda_n)(1-\tau-\delta\lambda_{n+1})\lambda_{n+1}^2\big]$$
$$+ \varepsilon^2\big[T_n^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_{n+1}^4\big].$$

Now, a simple computation shows that

$$T_n - (1+\tau+\delta\lambda_n)(1-\tau-\delta\lambda_{n+1})\lambda_{n+1}^2$$
$$= (\tau+\delta\lambda_{n+1})\Big[\frac{1+\tau}{2}(\lambda_{n+1}^2 - \lambda_n^2 - 4c_1(1+\tau)) + \frac{1+\tau}{2}(\lambda_{n+1}+\lambda_n)(\lambda_{n+1}-\lambda_n-4c_1\delta)$$
$$+ \delta\lambda_n\lambda_{n+1}(\lambda_{n+1}-\lambda_n-2c_1\delta)\Big],$$

which is positive, whenever (2.4.5) holds. Thus, $\mu^-_{n+1} - \gamma_n - c_1 > 0$ is positive, for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$.

Proof of $\mathbf{(2.6.4)_2}$ We have

$$\mu_n^- - \gamma_n + c_1 = \frac{Q_n + \varepsilon R_n - (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_n\sqrt{D_n}}{2\varepsilon(\tau+\delta\lambda_n)(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})},$$

where

$$Q_n = (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})\lambda_n,$$

$$R_n = \lambda_n^2(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1}) + (\tau+\delta\lambda_n)\Big[2c_1(1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1}) - \lambda_n^2(1+\tau+\delta\lambda_{n+1}) - \lambda_{n+1}^2(1+\tau+\delta\lambda_n)\Big].$$

When $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, we can see that the sign of

$Q_n + \varepsilon R_n - (1+\tau+\delta\lambda_n)(1+\tau+\delta\lambda_{n+1})\lambda_n\sqrt{D_n}$

is the same as that of

$(Q_n + \varepsilon R_n)^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_n^2 D_n$, and

$$\begin{aligned}
&(Q_n + \varepsilon R_n)^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_n^2 D_n \\
&= 2\varepsilon(1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})\lambda_n\big[R_n - (1+\tau+\delta\lambda_{n+1})(1-\tau-\delta\lambda_n)\lambda_n^2\big] \\
&\quad + \varepsilon^2\big[R_n^2 - (1+\tau+\delta\lambda_n)^2(1+\tau+\delta\lambda_{n+1})^2\lambda_n^4\big].
\end{aligned}$$

Also, a simple computation shows that

$$\begin{aligned}
&R_n - (1+\tau+\delta\lambda_{n+1})(1-\tau-\delta\lambda_n)\lambda_n^2 \\
&= (\tau+\delta\lambda_n)\Big[\frac{1+\tau}{2}(\lambda_n^2 - \lambda_{n+1}^2 + 4c_1(1+\tau)) + \frac{1+\tau}{2}(\lambda_n + \lambda_{n+1})(\lambda_n - \lambda_{n+1} + 4c_1\delta) \\
&\qquad + \delta\lambda_n\lambda_{n+1}(\lambda_n - \lambda_{n+1} + 2c_1\delta)\Big],
\end{aligned}$$

which is negative, whenever (2.4.5) holds. Thus, $\mu_n^- - \gamma_n + c_1 < 0$ is negative, for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$. The proof of (2.6.4) is completed. □

Theorem 2.7 *Let the assumptions of Theorem 2.6 hold. Then, there exists $M_4 > 0$ (independent of ϵ) such that,*

$$\mathrm{dist}_{\mathcal{U}_{d,\epsilon}}(\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,R}, (\mathfrak{M}^{\alpha,\mu})^{\sigma,\beta}) \leq M_4\sqrt{\varepsilon}, \quad \forall \varepsilon \in (0, \tilde{\varepsilon}(n)]. \tag{2.6.18}$$

Proof. Let $\mathbf{U}_0 = (\phi_0, u_0) \in \widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,R}$, then there exists a complete trajectory $\mathbf{U}_\varepsilon(t) = (\phi^\varepsilon(t), u^\varepsilon(t))_{t\in\mathbb{R}}$ lying in $\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon}$ and satisfies (2.4.4).

Let $(\phi(t))_{t\in\mathbb{R}}$ be a complete trajectory lying in $\mathfrak{M}^\alpha$. Then, there exists a complete trajectory $U(t) = (\phi(t), u(t)) \in (\mathfrak{M}^{\alpha,\mu})^{\sigma,\beta}, \quad \forall t \in \mathbb{R}$ which satisfies the non autonomous initial value problem (2.6.2). The function $\mathrm{Z}(t) = U(t) - \mathbf{U}_\epsilon(t)$ is defined for all time $t \in \mathbb{R}$ and satisfies the problem (2.6.5) and also satisfies (2.6.6).

Because $\mathcal{P}\mathcal{U}_d$ is a finite-dimensional subspace of $\mathcal{U}_d$, we can choose $U(0)$ such that $\mathrm{P}(0) = 0$. Then, proceeding like in the proof of Theorem (2.6), we obtain the upper semicontinuity estimate (2.6.18). ∎

CHAPTER 3

THE CAHN-HILLIARD EQUATION AS LIMIT OF A CONSERVED PHASE-FIELD SYSTEM

In this chapter, we consider again Problem (2.0.1), where $\delta \in [0,1]$ and $\varepsilon \in (0,1]$. We will investigate the convergence of the dynamics of (2.0.1) to those of the Cahn-Hilliard equation (1.2.8) as $(\varepsilon,\delta) \to (0,0)$.

In contrast with Chapter 2, we will need estimates that are independent of both parameters ε and δ. Recall that Problem (2.0.1) with $\delta = 0$ was studied in [5]. There, the proofs of the continuity properties of the global attractor, exponential attractors and inertial manifolds required the condition $g \in \mathcal{C}^8(\mathbb{R})$. Here, we will improve the results of [5] by weakening this condition to $g \in \mathcal{C}^4(\mathbb{R})$.

We will assume that in (1.5.3)-(1.5.6), $p \in [0,1]$ when $d = 3$.

3.1 A priori estimates

The estimate on the term $(Ng(\phi), \phi_t)$ in (2.1.19) is no longer suitable. Here, we have

$$(Ng(\phi), \phi_t) = (g'(\phi)N\phi, \phi_t) + (g''(\phi)|\nabla\phi|^2, \phi_t).$$

When $d = 1$, we have

$$\begin{aligned}
|(g'(\phi)N\phi, \phi_t)| &\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|N\phi\|\|\phi_t\| \\
&\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|\phi\|_2\|\phi_t\|, \\
|(g'(\phi)\nabla\phi, \nabla N\phi)| &\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|\|\nabla N\phi\| \\
&\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|\phi\|_1\|\phi\|_3,
\end{aligned}$$

and

$$\begin{aligned}
|(g''(\phi)|\nabla\phi|^2, \phi_t)| &\leq \|g''(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\|\nabla\phi\|\|\phi_t\| \\
&\leq c\|g''(\phi)\|_{L^\infty(\Omega)}\|\phi\|_1\|\phi\|_2\|\phi_t\|.
\end{aligned}$$

When $d = 2$, we have

$$\begin{aligned}
|(g'(\phi)N\phi, \phi_t)| &\leq c\left(\|\phi\|_{L^{4p+4}(\Omega)}^{p+1} + 1\right)\|N\phi\|_{L^4(\Omega)}\|\phi_t\| \\
&\leq c\left(\|\phi\|_1^{p+1} + 1\right)\|\phi\|_2^{1/2}\|\phi\|_3^{1/2}\|\phi_t\|, \\
|(g'(\phi)\nabla\phi, \nabla N\phi)| &\leq c\left(\|\phi\|_{L^{4p+4}(\Omega)}^{p+1} + 1\right)\|\nabla\phi\|_{L^4(\Omega)^2}\|\nabla N\phi\| \\
&\leq c\left(\|\phi\|_1^{p+1} + 1\right)\|\phi\|_2\|\phi\|_3,
\end{aligned}$$

and

$$\begin{aligned}
|(g''(\phi)|\nabla\phi|^2, \phi_t)| &\leq c\left(\|\phi\|_{L^{6p}(\Omega)}^{p} + 1\right)\|\nabla\phi\|_{L^6(\Omega)^2}^2\|\phi_t\| \\
&\leq c\left(\|\phi\|_1^{p} + 1\right)\|\phi\|_2^2\|\phi_t\|.
\end{aligned}$$

When $d = 3$, we have

$$
\begin{aligned}
|(g'(\phi)N\phi, \phi_t)| &\leq c\left(\|\phi\|^2_{L^\infty(\Omega)} + 1\right)\|N\phi\|\|\phi_t\| \\
&\leq c\left(\|\phi\|_1\|\phi\|_2 + 1\right)\|\phi\|_2\|\phi_t\|, \\
|(g'(\phi)\nabla\phi, \nabla N\phi)| &\leq c\left(\|\phi\|^2_{L^6(\Omega)} + 1\right)\|\nabla\phi\|_{L^6(\Omega)^3}\|\nabla N\phi\| \\
&\leq c\left(\|\phi\|^2_1 + 1\right)\|\phi\|_2\|\phi\|_3,
\end{aligned}
$$

and

$$
\begin{aligned}
|(g''(\phi)|\nabla\phi|^2, \phi_t)| &\leq c\left(\|\phi\|_{L^6(\Omega)} + 1\right)\|\nabla\phi\|^2_{L^6(\Omega)^3}\|\phi_t\| \\
&\leq c\left(\|\phi\|_1 + 1\right)\|\phi\|^2_2\|\phi_t\|.
\end{aligned}
$$

Hence from (2.1.19) on account of the estimates above, we deduce

$$
\begin{aligned}
&\frac{d}{dt}(\varepsilon\|\nabla u\|^2 + \|N\phi\|^2 + \tau\|\nabla\phi\|^2 + \delta\|N\phi\|^2) + \|\phi\|^2_3 + \|u\|^2_2 + \tau\|\phi_t\|^2 + 2\delta\|\nabla\phi_t\|^2 \\
&\leq M(t)(\|\phi\|^2_2 + 1)\|\phi\|^2_2 + c,
\end{aligned}
\tag{3.1.1}
$$

where

$$
M(t) = \begin{cases}
c\left(\|g'(\phi)\|^2_{L^\infty(\Omega)} + \|g''(\phi)\|^2_{L^\infty(\Omega)}\right), & \text{if } \ d = 1, \\
c\left(\|\phi\|^{4p+4}_1 + 1\right), & \text{if } \ d = 2, \\
c\left(\|\phi\|^4_1 + 1\right), & \text{if } \ d = 3.
\end{cases}
$$

We have the following estimates.

When $d = 1$, we have

$$
\begin{aligned}
&\|\nabla(g(\phi_1)-g(\phi_2))\|^2 \\
&\le 2\Big(2\sup_{\theta\in[0,1]}\|g''(\theta\phi_1+(1-\theta)\phi_2)\|^2_{L^\infty(\Omega)}\left(\|\nabla\phi_1\|^2+\|\nabla\phi_2\|^2\right)\|\phi\|^2_{L^\infty(\Omega)} \\
&\qquad +\sup_{\theta\in[0,1]}\|g'(\theta\phi_1+(1-\theta)\phi_2)\|_{L^\infty(\Omega)}\|\nabla\phi\|^2\Big) \\
&\le c\Big(\sup_{\theta\in[0,1]}\|g''(\theta\phi_1+(1-\theta)\phi_2)\|^2_{L^\infty(\Omega)}\left(\|\phi_1\|_1^2+\|\phi_2\|_1^2\right)\|\phi\|_1^2 \\
&\qquad +\sup_{\theta\in[0,1]}\|g'(\theta\phi_1+(1-\theta)\phi_2)\|_{L^\infty(\Omega)}\|\phi\|_1^2\Big).
\end{aligned}
\tag{3.1.2}
$$

When $d=2$, we have

$$
\begin{aligned}
&\|\nabla(g(\phi_1)-g(\phi_2))\|^2 \\
&\le c\left(1+\|\phi_1\|^{2p+2}_{L^{4p+4}(\Omega)}+\|\phi_2\|^{2p+2}_{L^{4p+4}(\Omega)}\right)\|\nabla\phi\|^2_{L^4(\Omega)^2} \\
&\quad +c\left(1+\|\phi_1\|^{2p}_{L^{6p}(\Omega)}+\|\phi_2\|^{2p}_{L^{6p}(\Omega)}\right)\left(\|\nabla\phi_1\|^2_{L^6(\Omega)^2}+\|\nabla\phi_2\|^2_{L^6(\Omega)^2}\right)\|\phi\|^2_{L^6(\Omega)} \\
&\le c\left(1+\|\phi_1\|_1^{2p+2}+\|\phi_2\|_1^{2p+2}\right)\|\phi\|_1\|\phi\|_2 \\
&\quad +c\left(1+\|\phi_1\|_1^{2p}+\|\phi_2\|_1^{2p}\right)\left(\|\phi_1\|_2^2+\|\phi_2\|_2^2\right)\|\phi\|_1^2.
\end{aligned}
\tag{3.1.3}
$$

When $d=3$, we obtain

$$
\begin{aligned}
&\|\nabla(g(\phi_1)-g(\phi_2))\|^2 \\
&\le c\left(1+\|\phi_1\|^4_{L^\infty(\Omega)}+\|\phi_2\|^4_{L^\infty(\Omega)}\right)\|\nabla\phi\|^2 \\
&\quad +c\left(1+\|\phi_1\|^2_{L^6(\Omega)}+\|\phi_2\|^2_{L^6(\Omega)}\right)\left(\|\nabla\phi_1\|^2_{L^6(\Omega)^3}+\|\nabla\phi_2\|^2_{L^6(\Omega)^3}\right)\|\phi\|^2_{L^6(\Omega)} \\
&\le c\left(1+\|\phi_1\|_1^2+\|\phi_2\|_1^2\right)\left(1+\|\phi_2\|_2^2+\|\phi_2\|_2^2\right)\|\phi\|_1^2 \\
&\quad +c\left(1+\|\phi_1\|_1^2+\|\phi_2\|_1^2\right)\left(\|\phi_1\|_2^2+\|\phi_2\|_2^2\right)\|\phi\|_1^2.
\end{aligned}
\tag{3.1.4}
$$

Instead of estimates (2.2.26), (2.2.27) and (2.2.28) in the proof of uniqueness in Theorem 2.1, here we make use of the estimates (3.1.2), (3.1.3) and (3.1.4) in

(2.2.25). Then, we deduce

$$\frac{d}{dt}\left[\tau\|\phi\|^2+(1+\delta)\|\phi\|_1^2+\varepsilon\|u\|^2\right]+\|\phi\|_2^2+\|u\|_1^2+\tau\|\phi_t\|_{-1}^2+2\delta\|\phi_t\|^2$$
$$\leq M_3(t)\|\phi\|_1^2, \tag{3.1.5}$$

where

$$M_3(t)=\begin{cases} c\sup_{\theta\in[0,1]}\|g'(\theta\phi_1+(1-\theta)\phi_2)\|_{L^\infty(\Omega)}^2 & \\ +c\sup_{\theta\in[0,1]}\|g''(\theta\phi_1+(1-\theta)\phi_2)\|_{L^\infty(\Omega)}^2\left(\|\phi_1\|_1^2+\|\phi_2\|_1^2\right), & \text{if} \quad d=1, \\ c\left(1+\|\phi_1\|_1^{4p+4}+\|\phi_2\|_1^{4p+4}\right)\left(\|\phi_1\|_2^2+\|\phi_2\|_2^2\right), & \text{if} \quad d=2, \\ c\left(1+\|\phi_1\|_1^2+\|\phi_2\|_1^2\right)\left(1+\|\phi_1\|_2^2+\|\phi_2\|_2^2\right), & \text{if} \quad d=3. \end{cases}$$

Applying the Gronwall's lemma to (3.1.5), we deduce that

$$\|(\phi(t),u(t))\|_{\mathcal{U}_{1,\varepsilon}}^2\leq ce^{\int_0^t M_3(s)ds}\|(\phi(0),u(0))\|_{\mathcal{U}_{1,\varepsilon}}^2,\ \forall t\geq 0, \tag{3.1.6}$$

hence the result.

3.2 The global attractor

On account of the *a priori estimates* in Section 3.1, Theorem 2.1 holds for all $\varepsilon\in(0,1]$ and $\delta\in[0,1]$. We can define the semigroup $S_{\varepsilon,\delta}(t)$ on $\mathcal{U}_1$ enjoying the same properties as $S_\varepsilon(t)$ with absorbing sets $\mathcal{B}_1$ and $\mathcal{B}_2$ in the form

$$\mathcal{B}_j=\{(\varphi,\psi)\in K_{\alpha,\sigma}\cap\mathcal{U}_j,\ \|(\varphi,\psi)\|_{\mathcal{U}_{j,\varepsilon}}\leq r_j\},\quad j=1,2,$$

where r_j is independent of ε and δ.

Similar to Theorem 2.2, we have the following result.

Theorem 3.1 *For every $\varepsilon \in (0,1]$ and every $\delta \in [0,1]$, the semigroup $S_{\varepsilon,\delta}(t)$, restricted to $K_{\alpha,\sigma}$, has the global attractor $\mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}$.*

The semigroup $S(t)$ generated by the unperturbed problem (the Cahn-Hilliard equation 1.2.8) possesses the global attractor $\mathcal{A}^{\alpha}$ on K_α (see [71]). We will show the upper semicontinuity of the global attractor $\mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}$ at $(\varepsilon,\delta) = (0,0)$.

Let us show the existence of a bounded absorbing set for $S_{\varepsilon,\delta}(t)$ in $\mathcal{U}_3 \cap K_{\alpha,\sigma}$ that will be useful in the proof of Theorem 3.2.

Lemma 3.1 *Let $\varepsilon \in (0,1]$ and $\delta \in [0,1]$. There exists $K > 0$ independent of ε and δ such that the solution $(\phi(t), u(t)) = S_{\varepsilon,\delta}(t)(\phi_0, u_0)$ satisfies*

$$\|\phi(t)\|_3^2 + \|u(t)\|_2^2 + \|\phi_t(t)\|_1^2 + \varepsilon\|u_t(t)\|^2 \leq K, \;\; \forall t \geq 2, \tag{3.2.1}$$

for any $(\phi_0, u_0) \in \mathcal{B}_2$.

Proof. Let $(\phi_0, u_0) \in \mathcal{B}_2$ and set $(\phi, u) = S_{\varepsilon,\delta}(t)(\phi_0, u_0)$. Since $\mathcal{B}_2$ is a bounded absorbing set for $S_{\varepsilon,\delta}(t)$ in $\mathcal{U}_2 \cap K_{\alpha,\sigma}$, there exists $c > 0$, independent of ε and δ, such that

$$\|\phi(t)\|_2^2 + \varepsilon\|u(t)\|_1^2 \leq c, \;\; \forall t \geq 0. \tag{3.2.2}$$

First, by differentiating the first and second equations of (2.0.1) with respect to time, we can show that the pair (ϕ_t, u_t) is a solution to the problem

$$\tau\phi_{tt} + N(\delta\phi_{tt} + N\phi_t + g'(\phi)\phi_t - u_t) = 0, \tag{3.2.3}$$

$$\varepsilon u_{tt} + \phi_{tt} + Nu_t = 0, \tag{3.2.4}$$

$$\phi_t|_{t=0} = \tilde{\mathrm{L}}\phi_0, \quad u_t|_{t=0} = -\frac{1}{\varepsilon}(\tilde{\mathrm{L}}\phi_0 + Nu_0), \tag{3.2.5}$$

where

$$\tilde{\mathrm{L}}\phi_0 = -(\tau I + \delta N)^{-1} N(N\phi_0 + g(\phi_0) - u_0). \tag{3.2.6}$$

Second, we also observe that $m(\tilde{\mathrm{L}}\phi_0) = 0$, thus $m(\phi_t(t)) = m(u_t(t)) = 0, \quad \forall t \geq 0$. We multiply (3.2.3) by $N^{-1}\phi_t$ and the second equation of (2.0.1) by u_t, then we integrate over Ω. We add the resulting equations to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2 + \|\nabla u\|^2\right) + \|\phi_t\|_1^2 + \varepsilon\|u_t\|^2 + (g'(\phi)\phi_t, \phi_t) = 0. \tag{3.2.7}$$

Owing to (1.5.5) and noting $\|\phi_t\|^2 \leq c\|\phi_t\|_{-1}\|\phi_t\|_1$, we deduce

$$\frac{d}{dt}\left(\tau\|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2 + \|\nabla u\|^2\right) + \|\phi_t\|_1^2 + 2\varepsilon\|u_t\|^2 \leq c\|\phi_t\|_{-1}^2. \tag{3.2.8}$$

Integrating (2.1.1) between 0 and ∞, we find that there exists $c > 0$, independent of ε and δ, such that $\int_0^\infty \left(\tau\|\phi_t(s)\|_{-1}^2 + \delta\|\phi_t(s)\|^2 + \|\nabla u\|^2\right) ds \leq c$. Now, applying the uniform Gronwall's lemma to (3.2.8) we obtain

$$\tau\|\phi_t(t)\|_{-1}^2 + \delta\|\phi_t(t)\|^2 + \|\nabla u(t)\|^2 \leq c, \quad \forall t \geq 1. \tag{3.2.9}$$

Then, integrating (3.2.8) between t and $t+1$, we infer

$$\int_t^{t+1} \left(\|\phi_t(s)\|_1^2 + \varepsilon\|u_t(s)\|^2\right) ds \leq c, \quad \forall t \geq 1. \tag{3.2.10}$$

We multiply (3.2.3) by $N^{-1}\phi_{tt}$ and (3.2.4) by u_t, and integrate over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi_t\|_1^2 + \tau\|\phi_{tt}\|_{-1}^2 + \delta\|\phi_{tt}\|^2 - (u_t, \phi_{tt}) + (N^{1/2}[g'(\phi)\phi_t], N^{-1/2}\phi_{tt}) = 0, \tag{3.2.11}$$

and

$$\frac{\varepsilon}{2}\frac{d}{dt}\|u_t\|^2 + (\phi_{tt}, u_t) + \|u_t\|_1^2 = 0. \tag{3.2.12}$$

Multiplying (3.2.3) by ϕ_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|\phi_t\|^2+\delta\|\phi_t\|_1^2\right)+\|\phi_t\|_2^2-(u_t,N\phi_t)+(g'(\phi)\phi_t,N\phi_t)=0. \qquad (3.2.13)$$

Summing (3.2.11), (3.2.12) and ϖ times (3.2.13), where $\varpi>0$ is small enough, we find after noting that $\|\nabla[g'(\phi)\phi_t]\|\le c\|\phi_t\|_1$

$$\frac{d}{dt}V(t)+\tau\|\phi_{tt}\|_{-1}^2+2\delta\|\phi_{tt}\|^2+\|u_t\|_1^2+\|\phi_t\|_2^2\le c\|\phi_t\|_1^2, \qquad (3.2.14)$$

where $V(t)=\|\phi_t\|_1^2+\varepsilon\|u_t\|^2+\varpi[\tau\|\phi_t\|^2+\delta\|\phi_t\|_1^2]$.

Applying the uniform Gronwall's lemma to (3.2.14), we find

$$\|\phi_t(t)\|_1^2+\varepsilon\|u_t(t)\|^2\le c,\quad \forall t\ge 2. \qquad (3.2.15)$$

Now, from the first and second equations of (2.0.1), we deduce that

$$\|\phi(t)\|_3^2\le c(\|\phi_t\|^2+\delta\|\phi_t\|_1^2+\|g(\phi)\|_1^2+\|u\|_1^2)\le c,\quad \forall t\ge 2 \qquad (3.2.16)$$

and

$$\|u(t)\|_2^2\le c(\varepsilon\|u_t\|^2+\|\phi_t\|^2)\le c.\quad \forall t\ge 2. \qquad (3.2.17)$$

Collecting (3.2.15), (3.2.16) and (3.2.17) yield the desired estimate (3.2.1). Consequently, the semigroup $S_{\varepsilon,\delta}(t)$ has an absorbing set in $\mathcal{U}_3\cap K_{\alpha,\sigma}$ of the form

$$\mathscr{B}_3=\{(\varphi,\psi)\in\mathcal{U}_3\cap K_{\alpha,\sigma},\ \|(\varphi,\psi)\|_{\mathcal{U}_3}\le\vartheta\},$$

where $\vartheta>0$ is independent of ε and δ. ∎

The global attractor $\mathcal{A}_{\varepsilon,\delta}^{\alpha,\sigma}$ satisfies $\mathcal{A}_{\varepsilon,\delta}^{\alpha,\sigma}\subset\mathscr{B}_3$, for all $\varepsilon\in(0,1]$ and $\delta\in[0,1]$.

Exploiting the invariance property of the attractor $S_{\varepsilon,\delta}(t)\mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta} = \mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}$, $\forall t \in \mathbb{R}$, we learn that, for any $z_0 = (\phi_0, u_0) \in \mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}$ there exists a complete trajectory $(\phi^{\varepsilon,\delta}(t), u^{\varepsilon,\delta}(t))_{t\in\mathbb{R}}$ in $\mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}$ such that $z^{\varepsilon,\delta}(t) = (\phi^{\varepsilon,\delta}(t), u^{\varepsilon,\delta}(t)) = S_{\varepsilon,\delta}(t)z_0$. In particular, there holds

$$(\phi^{\varepsilon,\delta}(t), u^{\varepsilon,\delta}(t))_{t\in\mathbb{R}} \subset \mathscr{B}_3,$$

for all $\varepsilon \in (0,1]$ and $\delta \in [0,1]$.

Now, we are in position to prove the following result.

Theorem 3.2 *There holds*

$$\lim_{(\varepsilon,\delta)\to(0,0)} \mathrm{dist}_{\mathcal{U}_2}(\mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}, (\mathcal{A}^{\alpha})^{\sigma}) = 0, \tag{3.2.18}$$

where $(\mathcal{A}^{\alpha})^{\sigma}$ is defined like in Section 2.3, with

$$\mathrm{L}_{\beta}\varphi = \beta + \frac{1}{1+\tau}(N\phi + \overline{g(\phi)}). \tag{3.2.19}$$

Proof. We follow [51] (see also [45]). The proof is based on a contradiction argument. We assume there exist $\varrho > 0$, sequences $\varepsilon_n, \delta_n \in (0,1]$, $\varepsilon_n \to 0$ and $\delta_n \to 0$, and a corresponding sequence $z_{0n} \in \mathcal{A}^{\alpha,\sigma}_{\varepsilon_n,\delta_n}$ such that

$$\mathrm{dist}_{\mathcal{U}_2}(z_{0n}, (\mathcal{A}^{\alpha})^{\sigma}) \geq \varrho. \tag{3.2.20}$$

We set

$$\mathcal{A}^{\alpha,\sigma} = \bigcup_{\varepsilon\in(0,1],\delta\in[0,1]} \mathcal{A}^{\alpha,\sigma}_{\varepsilon,\delta}.$$

From Lemma 3.1, we have that $\mathcal{A}^{\alpha,\sigma}$ is bounded in $\mathcal{U}_3$, hence it is relatively compact in $\mathcal{U}_2$. Let the complete trajectory $(\phi^n(t), u^n(t))_{t\in\mathbb{R}}$ such that $z_n(t) =$

$(\phi^n(t), u^n(t)) = S_{\varepsilon,\delta}(t) z_{0n}$. Due to the invariance property of the global attractor, it follows that

$$\bigcup_{t\in\mathbb{R}} \bigcup_{n\in\mathbb{N}} z_n(t) \subset \mathcal{A}^{\alpha,\sigma}$$

is a relatively compact set in $\mathcal{U}_2$. Hence, $\bigcup_{t\in\mathbb{R}} \bigcup_{n\in\mathbb{N}} (\phi^n(t), u^n(t))$ is a precompact set in $\mathcal{U}_2$, and the family of mappings $\{(\phi^n, u^n) \in \mathcal{C}([0,\infty); \mathcal{U}_2),\ n \geq 0\}$ is equicontinuous from $\mathbb{R}$ into $\mathcal{U}_2$. By Ascoli's theorem and a classical diagonalization method, it follows the existence of an element $(\tilde{\phi}, \tilde{u}) \in \mathcal{C}([0,\infty); \mathcal{U}_2)$ such that (up to a subsequence)

$$(\phi^n, u^n) \to (\tilde{\phi}, \tilde{u}) \text{ in } \mathcal{C}([-T,T]; \mathcal{U}_2),$$

for any $T > 0$, and $\sup\limits_{t\in\mathbb{R}} \|(\tilde{\phi}, \tilde{u})\|_{\mathcal{U}_2} < \infty$.

We have that

$$\tau \phi_t^n = -N(\delta_n \phi_t^n + N\phi^n + g(\phi^n) - u^n), \tag{3.2.21}$$

$$\phi_t^n = -\varepsilon_n u_t^n - N u^n. \tag{3.2.22}$$

Now, exploiting (3.2.1) we deduce that (up to a subsequence)

$$\phi_t^n \to \tilde{\phi}_t \quad \text{in} \quad \mathscr{D}'([-T,T]; V_1),$$

$$\tau \phi_t^n \to -N(N\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) \quad \text{in} \quad \mathcal{C}([-T,T]; V_{-2}),$$

and

$$\phi_t^n \to -N\tilde{u} \quad \text{in} \quad \mathcal{C}([-T,T]; V_{-1})$$

for any $T > 0$. Passing to the limit $n \to \infty$ in (3.2.21) and (3.2.22), and by uniqueness of limit, we find that

$$\tau\tilde{\phi}_t + N(N\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) = 0 \text{ in } \mathscr{D}'([-T,T];V_{-2}),$$

$$\tilde{\phi}_t + N\tilde{u} = 0 \qquad\qquad \text{in } \mathscr{D}'([-T,T];V_{-1}),$$

or

$$(1+\tau)\tilde{\phi}_t + N(N\tilde{\phi} + g(\tilde{\phi})) = 0 \text{ in } \mathscr{D}'([-T,T];V_{-2}),$$

for any $T > 0$. It follows that $\tilde{\phi}$ is a bounded complete trajectory of the semigroup $S(t)$. By characterization of the attractor it is clear that $\tilde{\phi}(0) \in \mathcal{A}^\alpha$, and by definition $(\tilde{\phi}(0), \mathrm{L}_{m(\tilde{u})}\tilde{\phi}(0)) \in (\mathcal{A}^\alpha)^\sigma$. Thus the convergence

$$z_{0n} \to (\tilde{\phi}(0), \mathrm{L}_{m(\tilde{u})}\tilde{\phi}(0)) \text{ in } \mathcal{U}_2$$

implies that

$$\lim_{n\to\infty} \mathrm{dist}_{\mathcal{U}_2}(z_{0n}, (\mathcal{A}^\alpha)^\sigma) = 0,$$

against the initial assumption. ∎

3.3 Inertial manifolds

In order to prove the existence of an inertial manifold for Problem (2.0.1), we rewrite the "prepared problem" (2.4.1) in the form:

$$U_t + \mathcal{A}U + (\tau I + \delta N)^{-1} N\mathbf{G}(U) = 0, \tag{3.3.1}$$

where $U = (\phi, u)$, $\mathbf{G}(U) = \left(-\mathfrak{g}(\phi), \frac{1}{\varepsilon}\mathfrak{g}(\phi)\right)$, $\mathfrak{g}$ is defined in (2.4.2) and $\mathcal{A}$ is given in Section 2.4.

Let $c_1 > 0$. There exists n such that $\lambda_n \geq 1$ and (cf. (2.4.5))

$$\lambda_{n+1} - \lambda_n > \max\left\{2c_1(1+\tau), 4c_1, \frac{c_1(1+\tau)}{\tau}\right\}. \tag{3.3.2}$$

Denote

$$D_k = \delta^2\lambda_k^2 + 2\delta(1+\tau-\varepsilon\lambda_k)\lambda_k + (1+\tau)^2 + 2\varepsilon(1-\tau)\lambda_k + \varepsilon^2\lambda_k^2,$$

$$\Delta_k = \frac{\lambda_{k+1}\sqrt{D_{k+1}}}{\tau+\delta\lambda_{k+1}} - \frac{\lambda_k\sqrt{D_k}}{\tau+\delta\lambda_k}, \qquad i_k^{\pm} = \frac{(1+\tau-\varepsilon\lambda_{k+1})\lambda_{k+1}^3}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{(1+\tau-\varepsilon\lambda_k)\lambda_k^3}{(\tau+\delta\lambda_k)^2},$$

$$g_k^{\pm} = \frac{\lambda_{k+1}^2}{\tau+\delta\lambda_{k+1}} \pm \frac{\lambda_k^2}{\tau+\delta\lambda_k}, \qquad h_k^{\pm} = \frac{\lambda_{k+1}^4}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_k^4}{(\tau+\delta\lambda_k)^2},$$

$$\sigma_n = (1+\tau+\varepsilon\lambda_n)^2 - 4\varepsilon\tau\lambda_n, \qquad x_k^{\pm} = \frac{\lambda_{k+1}\sqrt{\sigma_{n+1}}}{\tau+\delta\lambda_{k+1}} \pm \frac{\lambda_k\sqrt{\sigma_n}}{\tau+\delta\lambda_k},$$

$$f_k = \frac{(1+\tau+\varepsilon\lambda_{k+1})\lambda_{k+1}}{\tau+\delta\lambda_{k+1}} - \frac{(1+\tau+\varepsilon\lambda_k)\lambda_k}{\tau+\delta\lambda_k} - 2\varepsilon c_1(\lambda_k+\lambda_{k+1}),$$

$$j_k^{\pm} = \frac{[(1+\tau)^2 + 2\varepsilon(1-\tau)\lambda_{k+1} + \varepsilon^2\lambda_{k+1}^2]\lambda_{k+1}^2}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{[(1+\tau)^2 + 2\varepsilon(1-\tau)\lambda_k + \varepsilon^2\lambda_k^2]\lambda_k^2}{(\tau+\delta\lambda_k)^2}.$$

We now prove the following lemma.

Lemma 3.2 *Provided that n is large enough for (3.3.2) to hold, there exist $\varepsilon(n)$ and $\delta(n)$ suitably small such that the following inequalities are satisfied, for every $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$:*

(i) $f_n \geq 0$;

(ii) $\Delta_n \geq 0$;

(iii) $x_n^- < f_n < x_n^+$;

(iv) $f_n^4 - 2j_n^+ f_n^2 + (j_n^-)^2 < 0$;

(v) $j_n^+ - f_n^2 > 0$.

Proof. (i). The inequality $f_n \geq 0$ is equivalent to

$$\tau(1+\tau)(\lambda_{n+1}-\lambda_n)+\varepsilon(\lambda_{n+1}-\lambda_n)\big[\tau(\lambda_{n+1}+\lambda_n)+\delta\lambda_n\lambda_{n+1}\big]$$

$$-\,2\varepsilon c_1(\lambda_{n+1}+\lambda_n)(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})\geq 0,$$

which holds for every $\varepsilon\in(0,\varepsilon(n)]$.

(ii) We note that (2.4.8) is not always positive. Thus the inequality (2.4.6) holds for any $\delta\in[0,\delta(n)]$, $\varepsilon\in(0,\varepsilon(n)]$ and for some $\delta(n)>0$ and $\varepsilon(n)>0$. Hence (ii) holds.

(iii). Denote

$$a_n^{\pm}=\lambda_{n+1}^4\pm\lambda_n^4,\qquad\qquad b_n^{\pm}=(1-\tau)\left(\lambda_{n+1}^3\pm\lambda_n^3\right),$$

$$c_n^{\pm}=(1+\tau)^2(\lambda_{n+1}^2\pm\lambda_n^2),\qquad\qquad \alpha_n=(1+\tau)(\lambda_{n+1}-\lambda_n),$$

$$\beta_n=\lambda_{n+1}^2-\lambda_n^2-2c_1\tau(\lambda_{n+1}+\lambda_n)-2c_1\delta(\lambda_{n+1}+\lambda_n)^2$$

$$\varsigma_n=\lambda_{n+1}-\lambda_n-2c_1\delta(\lambda_{n+1}+\lambda_n).$$

These are all positive real numbers, whenever (3.3.2) is satisfied, except $b_n^{\pm}$ when $\tau>1$. The quantity x_n^- is positive when $\varepsilon\in(0,\varepsilon(n)]$.

The inequality $x_n^-<f_n$ is equivalent to

$$\tau\left[\alpha_n+\varepsilon\beta_n+(\lambda_n\sqrt{\sigma_n}-\lambda_{n+1}\sqrt{\sigma_{n+1}})\right]+\delta\lambda_{n+1}\lambda_n\left[\varepsilon\varsigma_n+(\sqrt{\sigma_n}-\sqrt{\sigma_{n+1}})\right]>0. \tag{3.3.3}$$

The terms $\lambda_{n+1}\sqrt{\sigma_{n+1}}-\lambda_n\sqrt{\sigma_n}$ and $\lambda_{n+1}^2\sigma_{n+1}-\lambda_n^2\sigma_n$ have the same sign, and

$$\lambda_{n+1}^2\sigma_{n+1}-\lambda_n^2\sigma_n=\varepsilon^2(\lambda_{n+1}^4-\lambda_n^4)+2\varepsilon(1-\tau)(\lambda_{n+1}^3-\lambda_n^3)+(1+\tau)^2(\lambda_{n+1}^2-\lambda_n^2),$$

which is positive when $\varepsilon\in(0,\varepsilon(n)]$. The inequality

$$\alpha_n+\varepsilon\beta_n+(\lambda_n\sqrt{\sigma_n}-\lambda_{n+1}\sqrt{\sigma_{n+1}})>0 \tag{3.3.4}$$

is equivalent to

$$\left(\varepsilon^2 a_n^- + 2\varepsilon b_n^- + c_n^-\right)^2 - 2\left(\alpha_n + \varepsilon\beta_n\right)^2\left(\varepsilon^2 a_n^+ + 2\varepsilon b_n^+ + c_n^+\right) + \left(\alpha_n + \varepsilon\beta_n\right)^4 < 0, \tag{3.3.5}$$

under the condition that

$$\varepsilon^2 a_n^+ + 2\varepsilon b_n^+ + c_n^+ - (\alpha_n + \varepsilon\beta_n)^2 > 0. \tag{3.3.6}$$

The inequality (3.3.6) holds whenever $\varepsilon \in (0, \varepsilon(n)]$. A computation of (3.3.5), observing that $\alpha_n^4 + (c_n^-)^2 - 2c_n^+\alpha_n^2 = 0$, gives the equivalent inequality

$$\begin{aligned}
&\varepsilon^3\left[(a_n^-)^2 - 2a_n^+\beta_n^2 + \beta_n^4\right] \\
&+ 4\varepsilon^2\left[\beta_n^3\alpha_n + a_n^- b_n^- - \beta_n\left(b_n^+\beta_n + a_n^+\alpha_n\right)\right] \\
&+ 2\varepsilon\left[3\beta_n^2\alpha_n^2 + a_n^- c_n^- + 2(b_n^-)^2 - \left(\beta_n^2 c_n^+ + 4b_n^+\alpha_n\beta_n + a_n^+\alpha_n^2\right)\right] \\
&+ 4\left[\beta_n\alpha_n^3 + b_n^- c_n^- - \left(\alpha_n\beta_n c_n^+ + b_n^+\alpha_n^2\right)\right] < 0.
\end{aligned} \tag{3.3.7}$$

We have

$$\begin{aligned}
&\beta_n\alpha_n^3 - \left(\alpha_n\beta_n c_n^+ + b_n^+\alpha_n^2\right) + b_n^- c_n^- \\
=& -4(1+\tau)^2(\lambda_{n+1}^2 - \lambda_n^2)\lambda_{n+1}\lambda_n\left[\tau(\lambda_{n+1} - \lambda_n - c_1(1+\tau)) - c_1\delta(1+\tau)(\lambda_n + \lambda_{n+1})\right],
\end{aligned} \tag{3.3.8}$$

which is negative, for every $\delta \in [0, \delta(n)]$, whenever (3.3.2) holds. The quadratic equation $y^2 - 2a_n^+ y + (a_n^-)^2 = 0$ has two positive roots $y^\pm = \left(\lambda_{n+1}^2 \pm \lambda_n^2\right)^2$ and

$$\begin{aligned}
&\beta_n^2 - y^- \\
&= -4c_1\tau(\lambda_{n+1} + \lambda_n)^2[\lambda_{n+1} - \lambda_n - c_1\tau] - 2c_1\delta(\lambda_{n+1} + \lambda_n)^3[\lambda_{n+1} - \lambda_n - 2c_1\tau] \\
&\quad + 4c_1^2\delta^2(\lambda_{n+1} + \lambda_n)^4,
\end{aligned}$$

which is negative for every $\delta \in [0, \delta(n)]$, whenever (3.3.2) holds. Therefore, $\beta_n^2 < y^- \leq y^+$.

Thus

$$(a_n^-)^2 - 2a_n^+\beta_n^2 + \beta_n^4 > 0,$$

hence (3.3.7) holds whenever $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$. It follows that (3.3.4) holds whenever $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$. Now, the inequality $\varepsilon\varsigma_n + \left(\sqrt{\sigma_n} - \sqrt{\sigma_{n+1}}\right)$ may be positive or negative; so that, (3.3.3) holds whenever $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$.

The inequality $f_n < x_n^+$ is equivalent to

$$\tau \left[\alpha_n + \varepsilon\beta_n - (\lambda_n\sqrt{\sigma_n} + \lambda_{n+1}\sqrt{\sigma_{n+1}})\right] + \delta\lambda_{n+1}\lambda_n \left[\varepsilon\varsigma_n - (\sqrt{\sigma_n} + \sqrt{\sigma_{n+1}})\right] < 0. \tag{3.3.9}$$

The inequalities

$$\alpha_n + \varepsilon\beta_n - (\lambda_n\sqrt{\sigma_n} + \lambda_{n+1}\sqrt{\sigma_{n+1}}) < 0 \quad \text{and} \quad \varepsilon\varsigma_n - (\sqrt{\sigma_n} + \sqrt{\sigma_{n+1}}) < 0$$

hold for every $\varepsilon \in (0, \varepsilon(n)]$. It follows that (3.3.9) holds.

(iv). The quadratic equation $x^2 - 2j_n^+x + (j_n^-)^2 = 0$ has two positive real roots which are $(x_n^\pm)^2$, hence (iv), due to (iii).

(v). A computation shows that

$$\begin{aligned}
& j_n^+ - f_n^2 \\
=& 2\frac{(1+\tau+\varepsilon\lambda_{n+1})(1+\tau+\varepsilon\lambda_n)\lambda_n\lambda_{n+1}}{(\tau+\delta\lambda_n)(\tau+\delta\lambda_{n+1})} - 4\tau\varepsilon\left(\frac{\lambda_n^3}{(\tau+\delta\lambda_n)^2} + \frac{\lambda_{n+1}^3}{(\tau+\delta\lambda_{n+1})^2}\right) \\
& + 4\varepsilon c_1 (\lambda_n + \lambda_{n+1}) \left(\frac{(1+\tau+\varepsilon\lambda_{n+1})\lambda_{n+1}}{\tau+\delta\lambda_{n+1}} - \frac{(1+\tau+\varepsilon\lambda_n)\lambda_n}{\tau+\delta\lambda_n} - \varepsilon c_1(\lambda_n + \lambda_{n+1})\right) \\
& \geq 0,
\end{aligned}$$

due to (i), whenever (3.3.2) is satisfied, hence (v) holds true for every $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$. The proof of the lemma is completed. ∎

We prove the following result.

Proposition 3.1 *For every $c_1 > 0$, there exist n (independent of δ and ε) and some positive real numbers $\delta(n)$, $\varepsilon(n)$ such that the spectral gap condition*

$$\mu_{n+1}^{-} - \mu_n^{-} > c_1 (\lambda_n + \lambda_{n+1}) \tag{3.3.10}$$

holds for every $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$.

Proof. It is clear that g_k^- is positive for every k. We have

$$\mu_{n+1}^{-} - \mu_n^{-} = \frac{1}{2\varepsilon}(f_n + \delta g_n^- - \Delta_n) + c_1 (\lambda_n + \lambda_{n+1}).$$

For every $\delta \in [0, \delta(n)]$ and $\varepsilon \in (0, \varepsilon(n)]$, the inequality $f_n - \Delta_n > 0$ is equivalent to

$$f_n^2 > \delta^2 h_n^+ + 2\delta i_n^+ + j_n^+ - 2\frac{\lambda_n \lambda_{n+1}\sqrt{D_n D_{n+1}}}{(\tau + \delta\lambda_n)(\tau + \delta\lambda_{n+1})},$$

which in turn, on account of (v) of Lemma 3.2, is equivalent to

$$\left(\delta^2 h_n^- + 2\delta i_n^- + j_n^-\right)^2 + f_n^4 - 2f_n^2\left(\delta^2 h_n^+ + 2\delta i_n^+ + j_n^+\right) < 0, \tag{3.3.11}$$

that is,

$$\begin{aligned} &\delta^4 (h_n^-)^2 + 4\delta^3 h_n^- i_n^- + 2\delta^2\left[2(i_n^-)^2 + h_n^- j_n^- - h_n^+ f_n^2\right] \\ &\quad + 4\delta\left(i_n^- j_n^- - i_n^+ f_n^2\right) + (j_n^-)^2 + f_n^4 - 2j_n^+ f_n^2 < 0. \end{aligned} \tag{3.3.12}$$

On account of (iv) of Lemma 3.2, the inequality (3.3.12) holds for every $\delta \in [0, \delta(n)]$, hence (3.3.10) holds true. ∎

We now prove the following result.

Theorem 3.3 *Let the assumptions of Theorem 2.4 hold. We assume that ε is dominated from above by δ, that is,*

$$\varepsilon, \delta \in (0,1], \quad \varepsilon \leq \xi\delta, \quad \text{for some } \xi \in (0,1]. \tag{3.3.13}$$

Then, there exists $\delta(n)$ such that, for every $\delta \in (0, \delta(n)]$, System (3.3.1) *has an inertial manifold $\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,\delta}$ (with dimensions independent of both ε and δ) in $K_{\alpha,\sigma}$.*

Proof. Let $X_n, Y_n, P, Q, X_{n1}, X_{n2}, \Psi_1, \Psi_2, \mathcal{P}, \mathcal{Q}, \mathcal{U}_1^1, (\mathcal{U}_1^1)^\perp, |||\cdot|||$ and $\langle\cdot,\cdot\rangle$, be as in Section 2.4, while Λ and Γ are defined in (2.4.3) and (2.4.24) respectively.

Noting that $2\varepsilon(u,v) \geq -\frac{2(1+\tau)}{2+\tau}\|u\|^2 - \frac{2+\tau}{2(1+\tau)}\varepsilon^2\|v\|^2$, we have

$$\begin{aligned}
\Psi_1(U,U) &= (1+\tau)\|u\|^2 + \delta\|\nabla u\|^2 + 2\varepsilon(u,v) + \varepsilon^2\|v\|^2 \\
&\geq \tau\frac{1+\tau}{2+\tau}\|u\|^2 + \delta\|\nabla u\|^2 + \frac{\tau}{2(1+\tau)}\varepsilon^2\|v\|^2, \\
&\geq c\left(\|u\|^2 + \delta\|u\|_1^2 + \varepsilon^2\|v\|^2\right), \quad \forall U \in X_n, \qquad (3.3.14) \\
\Psi_2(U,U) &= (1+\tau)\|u\|^2 + \delta\|\nabla u\|^2 + 2\varepsilon(u,v) + \varepsilon^2\|v\|^2 \\
&\geq c\left(\|u\|^2 + \delta\|u\|_1^2 + \varepsilon^2\|v\|^2\right), \quad \forall U \in Y_n. \qquad (3.3.15)
\end{aligned}$$

From (3.3.14) and (3.3.15), we can deduce that there exists c independent of ε and δ such that

$$|||U||| \geq c\left(\|u\| + \sqrt{\delta}\|u\|_1 + \varepsilon\|v\|\right), \quad \forall U = (u,v) \in \mathcal{U}_1, \tag{3.3.16}$$

for all $\delta \in (0, \delta(n)]$.

For any $U = (\phi, u), V = (\varphi, \psi) \in K_{\alpha,\sigma} \cap \mathcal{U}_d$, we have from (2.4.22) and (2.4.23) that

$$|||\Gamma\mathbf{G}(U)||| \leq c\left(\|\Gamma\mathfrak{g}(\phi)\| + \|\Gamma\nabla\mathfrak{g}(\phi)\|\right) \leq c\|\Lambda\mathfrak{g}(\phi)\| \leq c,$$

and $$|||\Gamma\mathbf{G}(U) - \Gamma\mathbf{G}(V)||| \le c\|\Gamma\mathfrak{g}(\phi) - \Gamma\mathfrak{g}(\varphi)\| + \sqrt{\delta}\|\Gamma\nabla\mathfrak{g}(\phi) - \Gamma\nabla\mathfrak{g}(\varphi)\|$$
$$\le c\|\Gamma\phi - \Gamma\varphi\| + \sqrt{\delta}\|\Lambda\phi - \Lambda\varphi\|$$
$$\le c|||\Gamma U - \Gamma V|||,$$

for all $\delta \in [0, \delta(n)]$, where $c > 0$ is independent of ε and δ. Thus, the nonlinear function $\mathbf{G}(U) : \mathcal{U}_d \to \mathcal{U}_d$ is globally Lipschitz continuous.

Moreover, there exist $C_8, C_9 > 0$, independent of ε and δ, such that

$$\|N(\tau I + \delta N)^{-1}\mathcal{Q}e^{s\mathcal{A}\mathcal{Q}}\|_{\mathcal{L}(\mathcal{Q}\mathcal{U}_d)} \le C_8\left(\frac{1}{(-s)^{1/2}} + \lambda_{n+1}\right)e^{s\mu^-_{n+1}}, \quad s < 0, \quad (3.3.17)$$

$$\|N(\tau I + \delta N)^{-1}\mathcal{P}e^{-s\mathcal{A}\mathcal{P}}\|_{\mathcal{L}(\mathcal{P}\mathcal{U}_d)} \le C_9\lambda_n e^{-s\mu^-_n}, \quad s \le 0, \quad (3.3.18)$$

for every $\delta \in (0, \delta(n)]$.

It follows from Theorem 1.3 that the semigroup $\tilde{S}_{\varepsilon,\delta}(t)$ generated by (3.3.1) admits an inertial manifold $\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,\delta}$ in $K_{\alpha,\sigma}$ of dimension n independent of ε, with respect to the metric induced by the norm $|||\Gamma\cdot|||$. Hence, there exists a Lipschitz mapping $\Phi^{\alpha,\sigma}_{\varepsilon,\delta} : K_{\alpha,\sigma} \cap \mathcal{P}\mathcal{U}_d \to \mathcal{Q}\mathcal{U}_d$ such that

$$\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,\delta} = \{\hat{\mathrm{p}} + \Phi^{\alpha,\sigma}_{\varepsilon,\delta}(\hat{\mathrm{p}}), \ \hat{\mathrm{p}} \in K_{\alpha,\sigma} \cap \mathcal{P}\mathcal{U}_d\}.$$

∎

Proof of (3.3.17) & (3.3.18): We can deduce that

$$\left(\frac{\lambda_k}{\tau + \delta\lambda_k}\right)^2 = \mu^-_k \frac{1 + \tau + (\delta + \varepsilon)\lambda_k + \sqrt{(1 + \tau + (\delta + \varepsilon)\lambda_k)^2 - 4\varepsilon\lambda_k(\tau + \delta\lambda_k)}}{2(\tau + \delta\lambda_k)^2},$$

hence

$$\left(\frac{\lambda_k}{\tau + \delta\lambda_k}\right)^2 \le \mu^-_k \frac{1 + \tau + (\delta + \varepsilon)\lambda_k}{(\tau + \delta\lambda_k)^2}. \quad (3.3.19)$$

There holds true

$$\frac{1+\tau+(\delta+\varepsilon)\lambda_k}{(\tau+\delta\lambda_k)^2} \leq K, \tag{3.3.20}$$

for every $\delta \in (0, \delta(k)]$, and where $K = \dfrac{2+\tau+\xi\tau}{\tau^2}$.

To see this, we just notice that (3.3.20) is equivalent to

$$K\tau^2 - 1 - \tau + (2K\tau\delta - \delta - \varepsilon)\lambda_k + K\delta^2\lambda_k^2 \geq 0,$$

which holds true whenever (due to (3.3.13))

$$K\tau^2 - 1 - \tau + (2K\tau - 1 - \xi)\delta\lambda_k + K\delta^2\lambda_k^2 \geq 0,$$

holds true.

It follows from (3.3.19) and (3.3.20) that

$$\frac{\lambda_k}{\tau+\delta\lambda_k} \leq K(\mu_k^{\pm})^{1/2}. \tag{3.3.21}$$

Also, we have that

$$\begin{aligned}
&\mu_k^- - \lambda_k^2 \\
&= \frac{\big[(1+\tau)\lambda_k + (\varepsilon+\delta)\lambda_k^2 - 2\varepsilon\lambda_k^2(\tau+\delta\lambda_k)\big]^2 - \lambda_k^2 D_k}{2\varepsilon(\tau+\delta\lambda_k)\big[(1+\tau)\lambda_k + (\varepsilon+\delta)\lambda_k^2 - 2\varepsilon\lambda_k^2(\tau+\delta\lambda_k) + \lambda_k\sqrt{D_k}\big]} \\
&= -\frac{2\lambda_k^2\big[\tau + (\varepsilon+\delta)\lambda_k - \varepsilon\lambda_k(\tau+\delta\lambda_k)\big]}{1+\tau+(\varepsilon+\delta)\lambda_k - 2\varepsilon\lambda_k(\tau+\delta\lambda_k) + \sqrt{D_k}} \\
&< 0,
\end{aligned} \tag{3.3.22}$$

for every $\varepsilon \in (0, \varepsilon(k)]$.

Let $\phi = \sum_{j=n+1}^{\infty} \alpha_j w_j$ be an element of $\mathcal{QE}$. We have, on account of (3.3.21),

$$\begin{aligned}
\|N(\tau I + \delta N)^{-1} \mathcal{Q} e^{s\mathcal{AQ}} \phi\|^2 &= \sum_{\mu_j^{\pm} \in \sigma_2} \left(\frac{\lambda_j}{\tau + \delta\lambda_j} e^{s\mu_j^{\pm}} \right)^2 \alpha_j^2 \\
&\leq K \sum_{\mu_j^{\pm} \in \sigma_2} \left((\mu_j^{\pm})^{1/2} e^{s\mu_j^{\pm}} \right)^2 \alpha_j^2 \\
&\leq K \sup_{\psi \geq \mu_{n+1}^{-}} \psi e^{2s\psi} \|\phi\|^2. \qquad (3.3.23)
\end{aligned}$$

Let

$$f(\psi) = \psi^{1/2} e^{s\psi},$$

we have $f'(\psi_0) = 0$, where $\psi_0 = -\dfrac{1}{2s}$.

If $\mu_{n+1}^{-} < \psi_0$, then

$$\sup_{\psi \geq \mu_{n+1}^{-}} f(\psi) \leq f(\psi_0) \leq \frac{1}{\sqrt{2}} (-s)^{-1/2} e^{s\mu_{n+1}^{-}}. \qquad (3.3.24)$$

If $\mu_{n+1}^{-} \geq \psi_0$, then

$$\sup_{\psi \geq \mu_{n+1}^{-}} f(\psi) \leq f(\mu_{n+1}^{-}). \qquad (3.3.25)$$

Inequality (3.3.17) follows from (3.3.23), due to (3.3.22), (3.3.24) and (3.3.25).

The other estimate is straightforward. Indeed, we have

$$\begin{aligned}
\|N(\tau I + \delta N)^{-1} \mathcal{P} e^{-s\mathcal{AP}} \phi\|^2 &= \sum_{j=0}^{n} \left(\frac{\lambda_j}{\tau + \delta\lambda_j} e^{-s\mu_j^{\pm}} \right)^2 \alpha_j^2 \\
&\leq \frac{1}{\tau^2} \lambda_n^2 \sup_{\psi \leq \mu_n^{-}} e^{-2s\psi} \|\phi\|^2 \\
&\leq C \lambda_n^2 e^{-2s\mu_n^{-}} \|\phi\|^2,
\end{aligned}$$

hence (3.3.18). □

3.4 A robust family of exponential attractors

The smoothing property (2.5.1), for the difference of two solutions to (2.0.1) also holds for $S_{\varepsilon,\delta}(t)$. The proof is similar to that of Proposition (2.4). Hence we have the following result (cf. proof of Theorem 2.5).

Theorem 3.4 *Assume that $g \in \mathcal{C}^3(\mathbb{R})$. Then, for every $\varepsilon \in (0,1]$ and every $\delta \in [0,1]$, the semigroup $S_{\varepsilon,\delta}(t)$ possesses an exponential attractor $\mathcal{E}^{\alpha,\sigma}_{\varepsilon,\delta}$ (with dimensions independent of both ε and δ) in $K_{\alpha,\sigma}$.*

Now, we show the existence of smooth absorbing sets.

Proposition 3.2 *In addition to* (1.5.3)-(1.5.6)*, we assume that $g \in \mathcal{C}^4(\mathbb{R})$. Then, for every $\varepsilon \in (0,1]$ and $\delta \in [0,1]$, the semigroup $S_{\varepsilon,\delta}(t)$ restricted to $K_{\alpha,\sigma}$ has an absorbing set of the form*

$$\mathcal{B}_4 = \{(\varphi,\psi) \in K_{\alpha,\sigma} \cap \mathcal{U}_4,\ \|(\varphi,\psi)\|_{\mathcal{U}_{4,\varepsilon}} \leq r_4\},$$

where r_4 is independent of ε and δ.

Proof. Let (ϕ_0, u_0) be in a bounded set B of $K_{\alpha,\sigma}$. We learnt from Section 3.2 that there exists $t_2 = t_2(B)$ (depending only on B) such that $\|(\phi(t), u(t))\|_{\mathcal{U}_{2,\varepsilon}} \leq r_2$, $\forall t \geq t_2$. Integrating (3.1.1) between t and $t+1$, we deduce that

$$\int_t^{t+1} \|(\phi(s), u(s))\|^2_{\mathcal{U}_{3,\varepsilon}} ds \leq c,\ \ \forall t \geq t_2. \tag{3.4.1}$$

We multiply the first equation of (2.0.1) by $N\phi_t$ and $N^2\phi$, the second one by N^2u, and integrate over Ω. Summing the resulting equations and noting that $\|N^{3/2}g(\phi)\| \leq c\|\phi\|_3, \quad \forall t \geq t_2$, we obtain

$$\frac{d}{dt}\left(\|N^{3/2}\phi\|^2 + \varepsilon\|Nu\|^2 + \tau\|N\phi\|^2 + \delta\|N^{3/2}\phi\|^2\right) + 2\|\phi\|_4^2$$

$$+ \|u\|_3^2 + \tau\|\phi_t\|_1^2 + 2\delta\|\phi_t\|_2^2 \leq c\left(\|\phi\|_3^2 + 1\right), \quad \forall t \geq t_2. \tag{3.4.2}$$

Applying the uniform Gronwall lemma to (3.4.2), on account of (3.4.1), there exists $t_3 = t_3(B)$ (depending only on B) such that

$$\|(\phi(t), u(t))\|_{\mathcal{U}_{3,\varepsilon}} \leq r_3, \ \ \forall t \geq t_3.$$

Then integrating (3.4.2) between t and $t+1$ we deduce that

$$\int_t^{t+1} \|(\phi(s), u(s))\|^2_{\mathcal{U}_{4,\varepsilon}} ds \leq c, \ \ \forall t \geq t_3. \tag{3.4.3}$$

Finally, we multiply the first equation of (2.0.1) by $N^2\phi_t$ and the second one by N^3u, and integrate over Ω. Summing the resulting equations and noting that $\|N^2 g(\phi)\| \leq c\|\phi\|_4, \quad \forall t \geq t_2$, we deduce

$$\frac{d}{dt}\left(\|N^2\phi\|^2 + \varepsilon\|N^{3/2}u\|^2\right) + \|u\|_4^2 + \tau\|\phi_t\|_2^2 + 2\delta\|\phi_t\|_3^2 \leq c\left(\|\phi\|_4^2 + 1\right), \ \forall t \geq t_2. \tag{3.4.4}$$

Applying the uniform Gronwall lemma to (3.4.4), due to (3.4.3), there exists $t_4 = t_4(B)$ (depending only on B) such that

$$\|(\phi(t), u(t))\|_{\mathcal{U}_{4,\varepsilon}} \leq r_4, \ \forall t \geq t_4,$$

hence the result. ∎

We have the following result on estimate of the difference of two solutions. We denote $S_{\varepsilon,\varepsilon} = S_\varepsilon$.

Proposition 3.3 *Let the assumption of Proposition 3.2 hold. Then, there exist $t_\star > 0$, c and c', all independent of ε and δ, such that*

$$\text{(i)} \quad \|S_{\varepsilon,\delta}(t)(\phi_0, u_0) - (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0)\|^2_{\mathcal{U}_{1,\varepsilon}} \le c(\varepsilon+\delta)^{1/4}e^{c't}, \ \ \forall t \ge t_\star, \tag{3.4.5}$$

for any $(\phi_0, u_0) \in \mathcal{B}_4$, any $\varepsilon \in (0,1]$ *and any* $\delta \in [0,1]$.

$$\text{(ii)} \quad \|S_\varepsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0)\|^2_{\mathcal{U}_1} \le c\sqrt[4]{\varepsilon}e^{c't}, \ \ \forall t \ge t_\star, \tag{3.4.6}$$

for any $(\phi_0, u_0) \in S_\varepsilon(1)\mathcal{B}_4$ and any $\varepsilon \in (0,1]$.

Proof. Let us take (ϕ_0, u_0) in $\mathcal{B}_4$. We set

$$(\phi_{\varepsilon,\delta}(t), u_{\varepsilon,\delta}(t)) = S_{\varepsilon,\delta}(t)(\phi_0, u_0), \quad (\phi(t), u(t)) = (S(t)\phi_0, \mathrm{L}_{m(u_0)}S(t)\phi_0).$$

The functions $\phi_{\varepsilon,\delta}$, $u_{\varepsilon,\delta}$ and ϕ satisfy the uniform estimates

$$\|\phi_{\varepsilon,\delta}(t)\|_4^2 + \|u_{\varepsilon,\delta}(t)\|_3^2 \le c, \quad \forall t \ge 0, \tag{3.4.7}$$

$$\|\phi(t)\|_4 \le c, \quad \forall t \ge 0. \tag{3.4.8}$$

We now set $P = \phi_{\varepsilon,\delta} - \phi$ and $R = u_{\varepsilon,\delta} - u$. The pair of functions (P, R) satisfies the following problem:

$$\tau P_t + N(\delta P_t + NP + g(\phi_{\varepsilon,\delta}) - g(\phi) - R) = -\delta N\phi_t, \tag{3.4.9}$$

$$\varepsilon R_t + P_t + NR = -\varepsilon u_t, \tag{3.4.10}$$

$$P|_{t=0} = 0, \quad R|_{t=0} = u_0 - \mathrm{L}_{m(u_0)}\phi_0. \tag{3.4.11}$$

We have that (the proof is similar to that of (2.5.26))

$$\int_0^t (\|\phi_t(s)\|_2^2 + \|u_t(s)\|^2) ds \le ce^{ct}, \ \ \forall t \ge 0. \tag{3.4.12}$$

Adopting same method as in the proof of Proposition 2.6 while relying on the estimates (3.4.12), (1.5.8) and (1.5.9), we reach

$$\begin{aligned} \|P(t)\|_1^2 + \varepsilon\|R(t)\|^2 &\le c_1 \left(\varepsilon\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|_1^2 + \varepsilon + \delta\right) \sqrt[4]{\varepsilon} e^{c't} \\ &\quad + c_2 \left[\varepsilon \left(\sqrt{\varepsilon}\|u_0 - \mathrm{L}_{m(u_0)}\phi_0\|_1 + \sqrt{\varepsilon + \delta}\right) + \delta^2\right] e^{c't}, \ \ \forall t > \sqrt{\varepsilon}, \end{aligned} \tag{3.4.13}$$

hence (i).

Like in the proof of Proposition 2.5, but relying on inequalities (1.5.8) and (1.5.9), there exists $c > 0$, independent of ε and δ, such that

$$\|S_{\varepsilon,\delta}(t)(\phi_0, u_0)\|_{\mathcal{U}_2} \le c, \quad \forall t \ge 1, \tag{3.4.14}$$

for every $\varepsilon \in (0, 1]$ and $\delta \in [0, 1]$. Finally, estimates (3.4.14) and (3.4.13) yield (ii). ∎

Now, we construct a robust family of exponential attractors which are both upper and lower semicontinuous at $\varepsilon = \delta = 0$.

We set $\widetilde{\mathcal{B}}_4 = S_{\varepsilon,\delta}(t^*)\mathcal{B}_4$, where $t^* > 0$ is independent of ε and δ (cf. proof of Theorem 2.5). We will denote $\mathcal{E}^{\alpha,\sigma}_{\varepsilon,\varepsilon} = \mathcal{E}^{\alpha,\sigma}_{\varepsilon}$.

Theorem 3.5 *Let the assumption of Proposition 3.2 hold. Then, there exist* $\varpi_1, \varpi_2 \in (0, \frac{1}{2}]$ *and* $M_1, M_2 > 0$*, all independent of* ε *and* δ*, and a family of exponential attractors* $\mathcal{E}^{\alpha,\sigma}_{\varepsilon,\delta}$ *enjoying all the properties of Theorem 3.4 and such* *that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{U}_{1,\varepsilon}}(\mathcal{E}^{\alpha,\sigma}_{\varepsilon,\delta},(\mathcal{E}^{\alpha})^{\sigma})\leq M_1(\varepsilon+\delta)^{\varpi_1}, \tag{3.4.15}$$

$$\mathrm{dist}_{\mathcal{U}_1}(\mathcal{E}^{\alpha,\sigma}_{\varepsilon},(\mathcal{E}^{\alpha})^{\sigma})\leq M_2\varepsilon^{\varpi_2}, \tag{3.4.16}$$

$$\text{and} \qquad \lim_{\varepsilon\to 0}\mathrm{dist}_{\mathcal{U}_1}((\mathcal{E}^{\alpha})^{\sigma},\mathcal{E}^{\alpha,\sigma}_{\varepsilon})=0, \tag{3.4.17}$$

where $\mathcal{E}^{\alpha}$ is an exponential attractor for $S(t)|_{K_\alpha}$.

Proof. On account of Theorem 1.2, we let $E_\varepsilon=\mathcal{U}_1$, $V_\varepsilon=\mathcal{U}_2$, $W_\varepsilon=\mathcal{U}_4$, $B_\varepsilon=\widetilde{\mathcal{B}}_4$. Assumptions 2-5 hold for $S_{\varepsilon,\varepsilon}(t)=S_\varepsilon(t)$ (cf. proof of Theorem 2.5). To verify Assumption 1, using the interpolation inequality, there exists a constant c such that for some $\theta\in[0,1]$ we have that for any φ and ψ in $\mathcal{B}$,

$$\begin{aligned}
\|\mathrm{L}_\beta\varphi-\mathrm{L}_\beta\psi\| &\leq c\big(\|\varphi-\psi\|_2+\|g(\varphi)-g(\psi)\|\big)\\
&\leq c\big(\|\varphi-\psi\|_2+\|g'(\theta\varphi+(1-\theta)\psi)\|_{L^\infty(\Omega)}\|\varphi-\psi\|\big)\\
&\leq c(\|\varphi-\psi\|_3^{1/2}+\|\varphi-\psi\|^{1/2})\|\varphi-\psi\|_1^{1/2}\\
&\leq c\|\varphi-\psi\|_1^{1/2}.
\end{aligned} \tag{3.4.18}$$

Hence, like in Theorem 2.5, we obtain the existence of exponential attractors on $\mathcal{U}_1$ that satisfy (3.4.16) and (3.4.17). Then, taking $B_\varepsilon=\mathcal{B}_4$ and relying on Estimate (3.4.5) instead of (3.4.6), we obtain the existence of a family of exponential attractors on $\overline{\mathcal{B}_4}^{\mathcal{U}_1}$ that satisfy (3.4.15) . ∎

3.5 Continuity of inertial manifolds

Now, we show continuity properties for the inertial manifolds $\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,\delta}$. Firstly, we note that there exists a Lipschitz mapping $\Phi^\alpha : PK_\alpha \cap \mathscr{D}(\Lambda) \to Q\mathscr{D}(\Lambda)$ such that the graph of Φ^α defines an inertial manifold

$$\mathfrak{M}^\alpha = \{p + \Phi^\alpha(p),\ \ p \in PK_\alpha \cap \mathscr{D}(\Lambda)\},$$

for the "prepared" Cahn-Hilliard equation 1.2.8 (see, e.g., [63]).

We prove the following result.

Theorem 3.6 *Let the assumptions of Theorem 3.3 hold. Let $\mu \in [-\alpha, \alpha]$ and $\beta \in [-\sigma, \sigma]$. Then, there exists $M_3 > 0$ and $M_4 > 0$ independent of ε such that,*

$$\mathrm{dist}_{\mathcal{U}_{d,\varepsilon}}((\mathfrak{M}^{\alpha,\mu}_R)^{\sigma,\beta}, \widetilde{\mathfrak{M}}^{\alpha,\sigma}_\varepsilon) \le M_3 \varepsilon^{1/2}, \tag{3.5.1}$$

$$\mathrm{dist}_{\mathcal{U}_{d,\epsilon}}(\widetilde{\mathfrak{M}}^{\alpha,\sigma}_{\varepsilon,R}, (\mathfrak{M}^{\alpha,\mu})^{\sigma,\beta}) \le M_4 \varepsilon^{1/2}, \quad \forall \varepsilon \in (0, \varepsilon(n)]. \tag{3.5.2}$$

Proof. The proof is similar to that of Theorem 2.6. Firstly, we can deduce from (3.3.14) through (3.3.16) that there exist c_1 and c_2 (independent of ε) such that

$$c_1 \|U\|_{\mathcal{U}_{d,\delta,\varepsilon^2}} \le |||\Gamma U||| \le c_2 \|U\|_{\mathcal{U}_{d,\varepsilon^2}}, \quad \forall U \in \mathcal{U}_d, \tag{3.5.3}$$

where

$$\|(\varphi, \psi)\|_{\mathcal{U}_{d,\delta,\varepsilon^2}} = \left(\|\varphi\|^2_{d-1} + \delta\|\varphi\|^2_d + \varepsilon^2\|\psi\|^2_{d-1}\right)^{1/2}.$$

Instead of (2.6.2), $U(t)$ satisfies the following problem:

$$U_t + \mathcal{A}U + (\tau I + \varepsilon N)^{-1} N\mathbf{G}(U) = (\tau I + \varepsilon N)^{-1} N\mathbf{F}(t),$$

$$U(0) = (\phi_0, \mathrm{L}_\beta \phi_0),$$

where

$$\mathbf{F}(t) = \big(\varepsilon\phi_t(t), (\tau I + \varepsilon N)N^{-1}u_t(t) - \phi_t(t)\big)\,.$$

We will prove that, there exists $M = M(n) > 0$, independent of ε, such that

$$\|\Lambda\phi_t(s)\| + \|\Gamma u_t(s)\| \leq Me^{-\gamma_n s}, \;\; \forall s \leq 0, \tag{3.5.4}$$

where

$$\gamma_n = \frac{\lambda_n^2 + \lambda_{n+1}^2}{2(1+\tau)}.$$

Also, for any $c_1 > 0$ and for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, there holds

$$\begin{cases} \mu_n^- - \gamma_n + c_1\lambda_n < 0, \\ \\ \mu_{n+1}^- - \gamma_n - c_1\lambda_{n+1} > 0. \end{cases} \tag{3.5.5}$$

The function $\mathrm{z}(t) = U(t) - U_\varepsilon(t)$ satisfies the equation $\mathrm{z}_t + \mathcal{A}\mathrm{z} + (\tau I + \varepsilon N)^{-1}N[\mathbf{G}(U) - \mathbf{G}(U_\varepsilon)] = (\tau I + \varepsilon N)^{-1}N\mathbf{F}(t)$.

Therefore,

$$\begin{aligned} \mathrm{z}(t) =& e^{-\mathcal{A}\mathcal{P}t}\mathrm{p}(0) + \int_0^t (\tau I + \varepsilon N)^{-1}Ne^{\mathcal{A}\mathcal{P}(s-t)}\mathcal{P}\left[\mathbf{G}(U_\varepsilon(s)) - \mathbf{G}(U(s)) + \mathbf{F}(s)\right]ds \\ &+ \int_{-\infty}^t (\tau I + \varepsilon N)^{-1}Ne^{\mathcal{A}\mathcal{Q}(s-t)}\mathcal{Q}\left[\mathbf{G}(U_\varepsilon(s)) - \mathbf{G}(U(s)) + \mathbf{F}(s)\right]ds, \end{aligned} \tag{3.5.6}$$

(cf., e.g., [22, 23, 71]). Like in Theorem (2.6), we can choose $U_\varepsilon(0)$ such that $\mathrm{p}(0) = 0$, and $\mathrm{z} \in \mathcal{C}_{\gamma_n}((-\infty, 0]; \mathcal{U}_d)$.

Thus, we deduce from (3.5.6) that

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&= \sup_{t\le 0} e^{\gamma_n t} \Bigg|\Bigg|\Bigg| \Bigg\{ \int_0^t (\tau I + \varepsilon N)^{-1} N e^{\mathcal{AP}(s-t)} \mathcal{P}[\Gamma\mathbf{G}(U_\varepsilon(s)) - \Gamma\mathbf{G}(U(s)) + \Gamma\mathbf{F}(s)] ds \\
&\qquad + \int_{-\infty}^t (\tau I + \varepsilon N)^{-1} N e^{\mathcal{AQ}(s-t)} \mathcal{Q}[\Gamma\mathbf{G}(U_\varepsilon(s)) - \Gamma\mathbf{G}(U(s)) + \Gamma\mathbf{F}(s)] ds \Bigg\} \Bigg|\Bigg|\Bigg| \\
&\le c_3 \sup_{t\le 0} \Bigg\{ \int_0^t \lambda_n e^{(\mu_n^- - \gamma_n)(s-t)} ds \\
&\qquad + \int_{-\infty}^t \left(\frac{1}{(-s)^{1/2}} + \lambda_{n+1} \right) e^{(\mu_{n+1}^- - \gamma_n)(s-t)} ds \Bigg\} \|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&+ c_2 \varepsilon \sup_{t\le 0} \Bigg\{ e^{\gamma_n t} \int_t^0 \lambda_n e^{(s-t)\mu_n^-} (\|\Lambda\phi_t(s)\| + \|\Gamma u_t(s)\|) ds \\
&\qquad + e^{\gamma_n t} \int_{-\infty}^t \left(\frac{1}{(-s)^{1/2}} + \lambda_{n+1} \right) e^{(s-t)\mu_{n+1}^-} (\|\Lambda\phi_t(s)\| + \|\Gamma u_t(s)\|) ds \Bigg\}.
\end{aligned} \tag{3.5.7}
$$

We have, for $a > 0$,

$$
\int_{-\infty}^t (-s)^{-1/2} e^{as} ds \le e^{at} \int_{-\infty}^0 (-s)^{-1/2} e^{as} ds,
$$

and

$$
\int_{-\infty}^0 (-s)^{-1/2} e^{as} ds = a^{-1/2} \int_0^\infty s^{-1/2} e^{-s} ds.
$$

We infer from (2.6.4), (3.5.4) and (3.5.7) that (cf. (2.6.8))

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&\le c_3 \left(\frac{\lambda_n}{\gamma_n - \mu_n^-} + \frac{\lambda_{n+1}}{\mu_{n+1}^- - \gamma_n} + \frac{1}{(\mu_{n+1}^- - \gamma_n)^{1/2}} \right) \|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \\
&\quad + c_2 \varepsilon \left(\frac{\lambda_n}{\gamma_n - \mu_n^-} + \frac{\lambda_{n+1}}{\mu_{n+1}^- - \gamma_n} + \frac{1}{(\mu_{n+1}^- - \gamma_n)^{1/2}} \right).
\end{aligned} \tag{3.5.8}
$$

We then deduce from (3.5.8) due to (3.5.5), that

$$\|\mathrm{z}\|_{\mathcal{C}_{\gamma_n}((-\infty,0];\mathcal{U}_d)} \leq M\varepsilon, \tag{3.5.9}$$

Hence we obtain (cf. (2.6.9) through (2.6.10))

$$|\!|\!|\Gamma(\phi_0, \mathrm{L}_\beta\phi_0) - \Gamma U_\varepsilon(0)|\!|\!| \leq M\varepsilon. \tag{3.5.10}$$

We can deduce from (3.5.3) that

$$c_4\|U\|_{\mathcal{U}_{d,\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}}\|U\|_{\mathcal{U}_{d,\varepsilon,\varepsilon^2}}, \quad \forall U \in \mathcal{U}_d. \tag{3.5.11}$$

Estimates (3.5.3), (3.5.10) and (3.5.11) eventually imply the lower semicontinuity estimate (3.5.1).

Finally, the proof of (3.5.1) is similar to that of (2.6.18). ▮

Proof of (3.5.4): In the proof of (2.6.3), taking $\delta = 0$, we have

$$\|\Lambda\phi_t(t)\| \leq c(n,R)e^{-\gamma_n t}, \quad \forall t \leq 0, \tag{3.5.12}$$

hence the first part of (3.5.4) holds.

Let us now prove the second part of (3.5.4). We remind that the unperturbed problem satisfies $\phi_t = Nu$, so that $\phi_{tt} = Nu_t$. We observe that $\tilde{\phi} = \phi_t$ is solution to the linearized problem:

$$\tilde{\phi}_t + A\tilde{\phi} + NF(t,\tilde{\phi}) = 0, \tag{3.5.13}$$

$$\tilde{\phi}(t)|_{t=0} = \tilde{\mathrm{L}}\phi_0, \tag{3.5.14}$$

where $\quad \tilde{\mathrm{L}}\phi_0 = -\dfrac{1}{1+\tau}N(N\phi_0 + \mathfrak{g}(\phi_0)), \quad F(t,\tilde{\phi}) = \dfrac{1}{1+\tau}\mathfrak{g}'(\phi(t))\tilde{\phi}.$

On account of (3.3.2), there exists a time varying $\mathcal{C}^1$-finite-dimensional invariant manifold $\mathfrak{M}_t^\alpha$ for problem (3.5.13) of the form (cf. Theorem 3.1, Chap. 3 of [24])

$$\mathfrak{M}_t^\alpha = \{\tilde{p} + \Phi^\alpha(t,\tilde{p}),\ \tilde{p} \in P\mathscr{D}(\Lambda)\},$$

where $\tilde{p}(t)$ is solution to:

$$\tilde{p}_t + A\tilde{p} + NPF(t, \tilde{p} + \Phi^\alpha(t, \tilde{p})) = 0, \tag{3.5.15}$$

$$\tilde{p}(0) = \tilde{p}_0. \tag{3.5.16}$$

We have, for all $t \in \mathbb{R}$,

$$\|\Lambda\Phi^\alpha(t, \tilde{p}(t))\| \leq c\left(1 + \|\Lambda\tilde{p}(t)\|\right), \tag{3.5.17}$$

$$\tilde{\phi}_t(t) = \tilde{p}_t(t) + (\Phi^\alpha(t, \tilde{p}(t)))'\tilde{p}_t(t), \tag{3.5.18}$$

$$\text{and} \quad \|\Lambda(\Phi^\alpha(t, \tilde{p}(t)))'\| \leq c. \tag{3.5.19}$$

On the one hand, we deduce from (3.5.15) and (3.5.17) that (cf. (2.6.14))

$$\|\Lambda\tilde{p}_t\| \leq c(n)\left(1 + \|\Lambda\tilde{p}\|\right). \tag{3.5.20}$$

On the other hand, we take the L^2-scalar product of (3.5.15) with $\Lambda^2\tilde{p}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda\tilde{p}\|^2 + \frac{1}{1+\tau}\|N\Lambda\tilde{p}\|^2 + \frac{1}{1+\tau}(P\Lambda F(t, \tilde{p} + \Phi^\alpha(t, \tilde{p})), N\Lambda\tilde{p}) = 0,$$

and deduce

$$-\|\Lambda\tilde{p}\|\frac{d}{dt}\|\Lambda\tilde{p}\| \leq \frac{1}{1+\tau}(\lambda_n^2 + c\lambda_n)\|\Lambda\tilde{p}\|^2 + c\lambda_n\|\Lambda\tilde{p}\|.$$

We have $\dfrac{1}{1+\tau}(\lambda_n^2 + c\lambda_n) < \gamma_n$ on account of (3.3.2).

Therefore,

$$-\|\Lambda\tilde{p}\|\frac{d}{dt}\|\Lambda\tilde{p}\| \leq \gamma_n\|\Lambda\tilde{p}\|^2 + c\lambda_n\|\Lambda\tilde{p}\|,$$

and then

$$\|\Lambda\tilde{p}(t)\| \leq c(n)\left(\|\Lambda\tilde{p}_0\| + 1\right)e^{-\gamma_n t}. \tag{3.5.21}$$

Note that, due to (3.5.12), we have $\|\Lambda\tilde{p}_0\| \leq c(n, R)$. Like (3.5.12) we can deduce

$$\|\Lambda\tilde{\phi}_t(t)\| \leq c(n, R)e^{-\gamma_n t}, \quad \forall t \leq 0. \tag{3.5.22}$$

In particular, $\|\Lambda\phi_{tt}(t)\| \le c(n,R)e^{-\gamma_n t}, \quad \forall t \le 0$, hence the second part of (3.5.4) holds. $\square$

Proof of $\mathbf{(3.5.5)_1}$ We have

$$\mu_{n+1}^{-} - \gamma_n - \lambda_{n+1}c_1 = \frac{(1+\tau)\lambda_{n+1} + \varepsilon S_n - \varepsilon^2 R_n - (1+\tau)\lambda_{n+1}\sqrt{D_{n+1}}}{2\varepsilon(1+\tau)(\tau+\delta\lambda_{n+1})},$$

where

$$S_n = \tau(\lambda_{n+1}^2 + \lambda_n^2) + 2\lambda_{n+1}^2 - 2c_1\tau(1+\tau)\lambda_{n+1},$$

$$R_n = \lambda_{n+1}(\lambda_{n+1}^2 + \lambda_n^2) + 2c_1(1+\tau)\lambda_{n+1}^2.$$

When $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, we can see that the sign of

$$(1+\tau)\lambda_{n+1} + \varepsilon S_n - \varepsilon^2 R_n - (1+\tau)\lambda_{n+1}\sqrt{D_{n+1}}$$

is the same as that of

$$2\varepsilon(1+\tau)^2\lambda_{n+1}\big[S_n - 2\lambda_{n+1}^2\big] + \varepsilon^2\big[(S_n - \varepsilon R_n)^2 - 2(1+\tau)^2\lambda_{n+1}R_n\big].$$

A simple computation shows that

$$\begin{aligned} S_n - 2\lambda_{n+1}^2 &= \tau\big[\lambda_{n+1}^2 - \lambda_n^2 - 2c_1(1+\tau)\lambda_{n+1}\big] \\ &\ge \tau(\lambda_{n+1}+\lambda_n)\big[\lambda_{n+1} - \lambda_n - 2c_1(1+\tau)\big], \end{aligned}$$

which is positive, whenever (3.3.2) holds. Thus, $\mu_{n+1}^{-} - \gamma_n - \lambda_{n+1}c_1 > 0$ is positive, for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$.

Proof of $\mathbf{(3.5.5)_2}$ We have

$$\mu_n^{-} - \gamma_n - \lambda_n c_1 = \frac{(1+\tau)\lambda_n + \varepsilon T_n - \varepsilon^2 Q_n - (1+\tau)\lambda_n\sqrt{D_n}}{2\varepsilon(1+\tau)(\tau+\delta\lambda_n)},$$

where

$$T_n = \tau(\lambda_n^2 + \lambda_{n+1}^2) + 2\lambda_n^2 - 2c_1\tau(1+\tau)\lambda_n,$$

$$Q_n = \lambda_n(\lambda_{n+1}^2 + \lambda_n^2) + 2c_1(1+\tau)\lambda_n^2.$$

When $\varepsilon \in (0, \tilde{\varepsilon}(n)]$, we can see that the sign of

$$(1+\tau)\lambda_n + \varepsilon T_n - \varepsilon^2 Q_n - (1+\tau)\lambda_n \sqrt{D_n}$$

is the same as that of

$$2\varepsilon(1+\tau)^2\lambda_n\big[T_n - 2\lambda_n^2\big] + \varepsilon^2\big[(T_n - \varepsilon Q_n)^2 - 2(1+\tau)^2\lambda_n Q_n\big].$$

A simple computation shows that

$$\begin{aligned} T_n - 2\lambda_n^2 &= \tau\big[\lambda_n^2 - \lambda_{n+1}^2 - 2c_1(1+\tau)\lambda_n\big] \\ &\le \tau(\lambda_n - \lambda_{n+1})\big[\lambda_{n+1} + \lambda_n - 2c_1(1+\tau)\big], \end{aligned}$$

which is negative, whenever (3.3.2) holds. Thus, $\mu_n^- - \gamma_n - \lambda_n c_1 > 0$ is positive, for every $\varepsilon \in (0, \tilde{\varepsilon}(n)]$.

CHAPTER 4

A CONSERVED PHASE-FIELD SYSTEM BASED ON THEORY OF HEAT CONDUCTION INVOLVING TWO TEMPERATURES

We now rewrite the system (1.2.12)-(1.2.13) as

$$\begin{cases} \tau\phi_t + N(\delta\phi_t + N\phi + g(\phi) - u) = 0, \\ \sigma u_t + \epsilon N u_t + \phi_t + Nu = 0, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \tag{4.0.1}$$

where $\delta > 0, \sigma > 0$ and $\epsilon \in (0, 1]$. g satisfies (1.5.3)-(1.5.6), with $p > 0$ arbitrary when $d = 1, 2$ and $p \in [0, 1]$ when $d = 3$.

4.1 Well-posedness

We multiply $(4.0.1)_1$ by $N^{-1}\phi_t$ and $(4.0.1)_2$ by u and we integrate over Ω, respectively. Summing the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\phi\|^2 + \epsilon\|\nabla u\|^2 + \sigma\|u\|^2 + 2\int_\Omega G(\phi)dx\right) + \|\nabla u\|^2$$
$$+ \tau\|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2 = 0. \tag{4.1.1}$$

Performing appropriate $L^2(\Omega)$ multiplications of equations $(4.0.1)_1$ and $(4.0.1)_2$, like in Section 2.1, there exist $c, c', c_1, c_2, c_3 > 0$ independent of ϵ such that

$$\frac{d}{dt}E(t) + c_0E(t) + c\left(\|u\|_1^2 + \|\phi\|_2^2 + \|\phi_t\|_{-1}^2 + \delta\|\phi_t\|^2\right) \leq c'. \tag{4.1.2}$$

where $c' = c(m(\phi_0), m(u_0))$, and

$$E(t) = (1 + \delta\varpi_2)\|\phi\|_1^2 + \epsilon\|u\|_1^2 + \sigma\|u\|^2 + \varpi_1\|\bar{\phi}\|_{-1}^2 + \delta\varpi_1\|\bar{\phi}\|^2$$
$$+ \tau\varpi_2\|\phi\|^2 + 2\int_\Omega G(\phi)dx.$$

satisfies

$$c_1\|(\phi(t), u(t))\|_{\mathcal{W}_{1,\epsilon}}^2 - c_2 \leq E(t) \leq c_3\left(\epsilon\|u(t)\|_1^2 + \|u(t)\|^2 + \|\phi(t)\|_1^{p+2} + 1\right).$$

Also, there exist $c_0, c_4 > 0$ independent of ϵ such that

$$\frac{d}{dt}\Upsilon(t) + \Upsilon(t) \leq c\big(\|\phi\|_2^2 + \|u\|^2 + \|\phi_t\|_{-1}^2\big). \tag{4.1.3}$$

where

$$\Upsilon(t) = \|N\phi\|^2 + \frac{\delta}{2}\|\nabla\phi\|^2 + \frac{\tau}{2}\|\phi\|^2 + \epsilon\|Nu\|^2 + \epsilon\|\nabla u\|^2 + \sigma\|\nabla u\|^2 + \sigma\|u\|^2.$$

and satisfies

$$c_0\|(\phi, u)\|^2_{\mathcal{W}_{2,\epsilon}} \leq \Upsilon(t) \leq c_1\|(\phi, u)\|^2_{\mathcal{W}_{2,\epsilon}}. \tag{4.1.4}$$

We have the following well-posedness result.

Theorem 4.1 *We assume that* (1.5.3)-(1.5.6) *hold. If* $(\phi_0, u_0) \in \mathcal{W}_1$, *then* (4.0.1) *possesses a unique solution* (ϕ, u) *such that*

$$(\phi, u) \in \mathcal{C}([0, T]; \mathcal{W}_1) \cap L^2(0, T; \mathcal{U}_2), \quad m(\phi(t)) = m(\phi_0), \;\; m(u(t)) = m(u_0),$$

for any $T > 0$. *Moreover, if* $(\phi_0, u_0) \in \mathcal{W}_2$, *then*

$$(\phi, u) \in \mathcal{C}([0, T]; \mathcal{W}_2) \cap L^2(0, T; \mathcal{U}_3).$$

Proof. (i) Existence: If $(\phi_0, u_0) \in \mathcal{W}_1$, then the approximate solutions (ϕ_m, u_m) are bounded independently of m (cf. (4.1.2)) and there exists a pair of functions (ϕ, u) such that $(\phi_m, u_m) \to (\phi, u)$ and $\phi \in L^\infty(0, T; V_1) \cap L^2(0, T; V_2)$, $u \in L^\infty(0, T; V_1) \cap L^2(0, T; V_1)$. Since g is continuous, we can pass to the limit as $m \to \infty$ in the approximate problem, and (ϕ, u) is solution to (4.0.1). Like in the proof of Theorem 2.1, from classical compactness theorems, we have $\phi \in \mathcal{C}([0, T]; V_2) \cap L^2(0, T; V_3)$ and $u \in \mathcal{C}([0, T]; V_2) \cap L^2(0, T; V_2)$.

(ii) Uniqueness: Let (ϕ_1, u_1) and (ϕ_2, u_2) be two solutions of (4.0.1). Setting $\phi = \phi_1 - \phi_2$ and $u = u_1 - u_2$, we have $\phi(0) = 0$, $u(0) = 0$, $m(\phi(t)) = 0$, $m(u(t)) = 0$, $\forall t \geq 0$, and (ϕ, u) satisfies

$$\tau\phi_t + N(\delta\phi_t + N\phi + g(\phi_1) - g(\phi_2) - u) = 0, \tag{4.1.5}$$

$$\sigma u_t - \epsilon\Delta u_t + \phi_t + Nu = 0. \tag{4.1.6}$$

Again, proceeding like in the proof of uniqueness in Theorem 2.1, we obtain

$$\|(\phi(t), u(t))\|^2_{\mathcal{W}_{1,\epsilon}} \le ce^{ct}\|(\phi(0), u(0))\|^2_{\mathcal{W}_{1,\epsilon}}, \; \forall t \ge 0, \tag{4.1.7}$$

hence the result. ∎

4.2 The global attractor

Thanks to Theorem 4.1, we can define the semigroup

$$S_\epsilon(t) : \mathcal{W}_1 \to \mathcal{W}_1, \quad (\phi_0, u_0) \mapsto (\phi(t), u(t)), \quad t \ge 0,$$

where $(\phi(t), u(t))$ is the solution to (4.0.1) at time t. The semigroup $S_\epsilon(t)$ is strongly continuous. We apply Gronwall's lemma to (4.1.2) and we deduce the existence of an absorbing set for $S_\epsilon(t)$ on $K_{\alpha,\rho}$ of the form

$$\mathcal{B}_1 = \{(\varphi, \psi) \in \widetilde{K}_{\alpha,\rho}, \; \|(\varphi, \psi)\|_{\mathcal{W}_{1,\epsilon}} \le r_1\},$$

where r_1 is independent of ϵ.

To prove the existence of a compact global attractor for $S_\epsilon(t)$, we will perform a splitting of the semigroup $S_\epsilon(t)$. Given $(\phi_0, u_0) \in \mathcal{B}_1$, we start by splitting the solution (ϕ, u) of (4.0.1) into

$$(\phi, u) = (\varphi, v) + (\psi, w)$$

where (φ, v) and (ψ, w) solve, respectively, the problems

$$\begin{cases} \tau\varphi_t + N(\delta\varphi_t + N\varphi + g(\phi) - v) + \varphi = \phi, \\ \sigma v_t + \epsilon N v_t + \varphi_t + Nv + v = u, \\ \varphi(0) = 0, \quad \epsilon v(0) = 0, \end{cases} \tag{4.2.1}$$

and

$$\begin{cases} \tau\psi_t + N(\delta\psi_t + N\psi - w) + \psi = 0, \\ \sigma w_t + \epsilon N w_t + \psi_t + Nw + w = 0, \\ \psi(0) = \phi_0, \quad \epsilon w(0) = \epsilon u_0, \end{cases} \tag{4.2.2}$$

Computations show that

$$\begin{aligned} m(\psi(t)) &= m(\phi_0)e^{-\frac{1}{\tau}t}, \\ m(w(t)) &= m(u_0)e^{-\frac{1}{\sigma}t} + \frac{1}{\tau - \sigma} m(\phi_0)\big(e^{-\frac{1}{\tau}t} - e^{-\frac{1}{\sigma}t}\big), \\ m(\varphi(t)) &= m(\phi_0)\big(1 - e^{-\frac{1}{\tau}t}\big), \\ m(v(t)) &= m(u_0)\big(1 - e^{-\frac{1}{\sigma}t}\big) - \frac{1}{\tau - \sigma} m(\phi_0)\big(e^{-\frac{1}{\tau}t} - e^{-\frac{1}{\sigma}t}\big). \end{aligned} \tag{4.2.3}$$

Throughout the following two lemmas given below, the generic constant c is independent of ϵ but may depend on the size of $\mathcal{B}_1$, (but not on any components of the initial data namely ϕ_0 and u_0). Moreover, all other constants are independent of ϵ, u_0, ϕ_0 and $\mathcal{B}_1$.

Lemma 4.1 *For every $t \geq 0$ and some $\nu > 0$,*

$$\|(\psi(t), w(t))\|_{\mathcal{W}_{1,\epsilon}} \leq ce^{-\nu t}. \tag{4.2.4}$$

Proof. We multiply $(4.2.2)_1$ by $N^{-1}\bar{\psi}_t$ and $(4.2.2)_2$ by $\bar{w}$, integrate over Ω, and then add the resulting equations to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla\psi\|^2+\|\bar{\psi}\|^2+\epsilon\|\nabla w\|^2+\sigma\|\bar{w}\|^2\big)+\tau\|N^{-1/2}\bar{\psi}_t\|_{-1}^2+\delta\|\bar{\psi}_t\|^2\\&+\|\nabla w\|^2+\|\bar{w}\|^2=0.\end{aligned}\tag{4.2.5}$$

Next, we multiply $(4.2.2)_1$ by $N^{-1}\bar{\psi}$ and integrate over Ω, to get

$$\frac{1}{2}\frac{d}{dt}\big[\tau\|N^{-1/2}\bar{\psi}\|^2+\delta\|\bar{\psi}\|^2\big]+\|\nabla\psi\|^2+\|N^{-1/2}\bar{\psi}\|^2-(w,\bar{\psi})=0.\tag{4.2.6}$$

Now, we sum (4.2.5) and ϖ(4.2.6), for $\varpi>0$ to be appropriately chosen later, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_1(t)+\tau\|N^{-1/2}\bar{\psi}_t\|^2+\delta\|\bar{\psi}_t\|^2+\|\nabla w\|^2+\|\bar{w}\|^2+\varpi\|\nabla\psi\|^2\\&+\varpi\|N^{-1/2}\bar{\psi}\|^2-\varpi(w,\bar{\psi})=0,\end{aligned}\tag{4.2.7}$$

where

$$E_1(t)=\|\nabla\psi\|^2+\epsilon\|\nabla w\|^2+\sigma\|\bar{w}\|^2+\varpi\tau\|N^{-1/2}\bar{\psi}\|^2+\varpi\delta\|\bar{\psi}\|^2.$$

We observe that $-\varpi(w,\bar{\psi})\geq-\frac{\varpi}{2}\|\nabla w\|^2-\frac{\varpi}{2}\|N^{-1/2}\bar{\psi}\|^2$, and that there exists $c'>0$ such that

$$\begin{aligned}c'E_1(t)\leq&\tau\|N^{-1/2}\bar{\psi}_t\|^2+\delta\|\bar{\psi}_t\|^2+\left(1-\frac{\varpi}{2}\right)\|\nabla w\|^2+\|\bar{w}\|^2+\varpi\|\nabla\psi\|^2\\&+\frac{\varpi}{2}\|N^{-1/2}\bar{\psi}\|^2\end{aligned}$$

On account of the above last two inequalities, (4.2.7) gives

$$\frac{d}{dt}E_1(t)+2c'E_1(t)\leq 0,$$

then simple integration over $(0,t)$ gives

$$E_1(t)\leq E_1(0)e^{-c't},\quad\forall t\geq 0.\tag{4.2.8}$$

There exists $C_0 > 0$ such that for every $t \geq 0$,

$$\|(\psi(t), w(t))\|^2_{\mathcal{W}_{1,\epsilon}} - |m(\psi)|^2 - (\epsilon + \sigma)|m(w)|^2 \leq E_1(t) \leq C_0\|(\psi(t), w(t))\|^2_{\mathcal{W}_{1,\epsilon}}. \tag{4.2.9}$$

From the inequality (4.2.9), (4.2.8) together with $(4.2.3)_1$ and $(4.2.3)_2$ we obtain (4.2.4). ∎

Lemma 4.2 *For every $t \geq 0$,*

$$\|(\varphi(t), v(t))\|^2_{\mathcal{W}_{2,\epsilon}} \leq c. \tag{4.2.10}$$

Proof. We multiply $(4.2.1)_1$ by φ_t and φ, then we multiply $(4.2.1)_2$ by Nv, we integrate them over Ω, summing the resulting equations we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|N\varphi\|^2 + (1+\tau)\|\varphi\|^2 + \delta\|\nabla\varphi\|^2 + \epsilon\|Nv\|^2 + \sigma\|\nabla v\|^2\big) + \tau\|\varphi_t\|^2 + \delta\|\nabla\varphi_t\|^2 \\
&\quad + \|N\varphi\|^2 + \|\varphi\|^2 + \|Nv\|^2 + \|\nabla v\|^2 \\
&= -(\nabla g(\phi), \nabla\varphi_t) - (\nabla g(\phi), \nabla\varphi) + (\phi, \varphi_t) + (u, Nv) + (Nv, \varphi) + (\phi, \varphi),
\end{aligned} \tag{4.2.11}$$

Using Young's inequality, the following hold

$$\begin{aligned}
&|(\phi, \varphi_t)| \leq \frac{1}{2\tau}\|\phi\|^2 + \frac{\tau}{2}\|\varphi_t\|^2, \qquad |(u, Nv)| \leq \frac{3}{4}\|u\|^2 + \frac{1}{3}\|Nv\|^2, \\
&|(Nv, \varphi)| \leq \frac{3}{4}\|\varphi\|^2 + \frac{1}{3}\|Nv\|^2, \qquad |(\phi, \varphi)| \leq 2\|\phi\|^2 + \frac{1}{8}\|\varphi\|^2, \\
&|(\nabla g(\phi), \nabla\varphi_t)| \leq c\|\phi\|_2^2 + \frac{\delta}{2}\|\nabla\varphi_t\|, \\
&|(\nabla g(\phi), \nabla\varphi)| \leq c\|\phi\|_2^2 + \frac{1}{4}\|\nabla\varphi\|
\end{aligned} \tag{4.2.12}$$

On account of (4.2.12), we get that (4.2.11) yields

$$\frac{1}{2}\frac{d}{dt}E_2(t) + \frac{\tau}{2}\|\varphi_t\|^2 + \frac{\delta}{2}\|\nabla\varphi_t\|^2 + \frac{1}{2}\|N\varphi\|^2 + \frac{1}{4}\|\nabla\varphi\|^2 + \frac{1}{8}\|\varphi\|^2$$
$$+ \frac{1}{3}\|Nv\|^2 + \|\nabla v\|^2 \leq c(\|\phi\|_2^2 + \|u\|^2), \tag{4.2.13}$$

where

$$E_2(t) = \|N\varphi\|^2 + (1+\tau)\|\varphi\|^2 + \delta\|\nabla\varphi\|^2 + \epsilon\|Nv\|^2 + \sigma\|\nabla v\|^2.$$

Clearly, there exists $\varrho > 0$ such that

$$\varrho E_2(t) \leq \frac{1}{2}\|N\varphi\|^2 + \frac{1}{4}\|\nabla\varphi\|^2 + \frac{1}{8}\|\varphi\|^2 + \frac{1}{3}\|Nv\|^2 + \|\nabla v\|^2.$$

Hence, we deduce from (4.2.13) that

$$\frac{d}{dt}E_2(t) + \varrho E_2(t) \leq c(\|\phi\|_2^2 + \|u\|^2). \tag{4.2.14}$$

From (4.1.2) we obtain that $\int_t^{t+1}(\|\phi(s)\|_2^2 + \|u(s)\|^2)ds \leq C$, thus applying lemma (1.6) to (4.2.14) we obtain at once that $E_2(t) \leq C, \quad \forall t \geq 0$, then it follows that

$$\|\varphi\|_2^2 + \epsilon\|v\|_2^2 + \sigma\|v\|_1^2 \leq C + (\epsilon+\sigma)|m(v)|^2 + |m(\varphi)|^2.$$

Hence (4.2.10). ∎

Theorem 4.2 *For every $\epsilon \in (0,1]$, the semigroup $S_\epsilon(t)$ has the global attractor $\mathcal{A}_\epsilon^{\alpha,\rho}$ in $\widetilde{K}_{\alpha,\rho}$.*

Proof. From Lemma 4.1 and Lemma 4.2, we get that $S_\epsilon(t)\mathcal{B}_1$ is (exponentially) attracted by a closed bounded subset $\mathscr{B}_2 \subset \mathcal{W}_2 \cap \widetilde{K}_{\alpha,\rho}$. That is to say $\mathscr{B}_2$ is a compact attracting set. Thus, by Theorem 1.1 (see also [58, 71]), the semigroup $S_\epsilon(t)$ for each $\epsilon \in (0,1]$ possesses a compact global attractor $\mathcal{A}_\epsilon^{\alpha,\rho} \subset \mathscr{B}_2$. ∎

4.2.1 Upper semicontinuity of the global attractor

First, we prove the existence of a bounded absorbing set on $\mathcal{W}_2 \cap \widetilde{K}_{\alpha,\rho}$. Let $R_2 > 0$ such that $\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}} \leq R_2$. Integrating (4.1.2) over $(t, t+1)$, we obtain that

$$\int_t^{t+1} \big(\|\phi(s)\|_2^2 + \|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2\big) ds \leq c.$$

Application of the generalized Gronwall's lemma 1.6 to (4.1.3) yields

$$\Upsilon(t) \leq C\Upsilon(0)e^{-\nu t} + K_1, \quad \forall t \geq 0.$$

It follows, due to (4.1.4), that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}} \leq K_2 e^{-\nu t} + K_1, \quad \forall t \geq 0, \tag{4.2.15}$$

where the positive constants $K_2 = K_2(R_2)$ and $K_1 = K_1(r_1)$ are independent of ϵ. Hence the existence of a bounded absorbing set on $\mathcal{W}_2 \cap \widetilde{K}_{\alpha,\rho}$ of the form

$$\mathcal{B}_2 = \{(\varphi, \psi) \in \mathcal{W}_2 \cap \widetilde{K}_{\alpha,\rho},\ \|(\varphi, \psi)\|_{\mathcal{W}_{2,\epsilon}} \leq r_2\},$$

where r_2 is independent of ϵ.

The semigroup $S_0(t)$ generated by the unperturbed problem (2.0.1) possesses the global attractor $\mathcal{A}_0^{\alpha,\rho}$ on $K_{\alpha,\rho}$ (cf. Theorem 2.2). We have the following result.

Theorem 4.3 *There holds*

$$\lim_{\epsilon \to 0} \mathrm{dist}_{\mathcal{U}_1}(\mathcal{A}_\epsilon^{\alpha,\rho}, \mathcal{A}_0^{\alpha,\rho}) = 0. \tag{4.2.16}$$

Proof. Like in Lemma 3.1, we can show that there exists $K > 0$ independent of ϵ, such that the solution $(\phi, u) = S_\epsilon(t)(\phi_0, u_0)$ satisfies

$$\|\phi(t)\|_4^2 + \|u(t)\|_2^2 + \|\phi_t(t)\|_2^2 + \|u_t(t)\|_1^2 + \epsilon\|u_t(t)\|_2^2 \leq K, \ \ \forall t \geq 2, \tag{4.2.17}$$

for any $(\phi_0, u_0) \in \mathcal{B}_2$. Then we obtain the result by proceeding like in the proof of Theorem 3.2. ∎

4.3 Inertial manifolds

In this section, the assumptions on d and Ω are as in Section 2.4. Moreover, $\mathfrak{g}, \Lambda, \Gamma$ are also the same as given in Section 2.4.

We introduce the "prepared problem":

$$\begin{cases} \tau\phi_t + N(\delta\phi_t + N\phi + \mathfrak{g}(\phi) - u) = 0, \\ \sigma u_t + \epsilon N u_t + \phi_t + Nu = 0, \end{cases} \tag{4.3.1}$$

which can be rewritten in the following form:

$$U_t + \mathbb{A}U + \mathbb{G}(U) = 0, \tag{4.3.2}$$

for every $\epsilon \in (0, 1]$, where $U = (\phi, u)$,

$$\mathbb{G}(U) = \left((\tau I + \delta N)^{-1} N\mathfrak{g}(\phi), -(\sigma I + \epsilon N)^{-1}(\tau I + \delta N)^{-1} N\mathfrak{g}(\phi)\right),$$

and

$$\mathbb{A} = \begin{pmatrix} (\tau I + \delta N)^{-1} N^2 & -(\tau I + \delta N)^{-1} N \\ -(\sigma I + \epsilon N)^{-1}(\tau I + \delta N)^{-1} N^2 & (\sigma I + \epsilon N)^{-1}((\tau I + \delta N)^{-1} N + N) \end{pmatrix}.$$

The operator $\mathbb{A} : \mathcal{W}_3 \to \mathcal{W}_1$ is non self-adjoint, and has a positive spectrum

$$\begin{aligned}\eta_k^{\pm} = & \frac{\lambda_k(1+\tau+(\delta+\sigma)\lambda_k+\epsilon\lambda_k^2)}{2(\sigma+\epsilon\lambda_k)(\tau+\delta\lambda_k)} \\ & \pm \frac{\lambda_k\sqrt{(1+\tau+(\delta+\sigma)\lambda_k+\epsilon\lambda_k^2)^2-4\sigma\lambda_k(\tau+\delta\lambda_k)-4\epsilon\lambda_k^2(\tau+\delta\lambda_k)}}{2(\sigma+\epsilon\lambda_k)(\tau+\delta\lambda_k)},\end{aligned}$$

for every $\sigma > 0$ and for every $\epsilon \in (0, 1]$, for $k = 0, 1, 2, ...$

Indeed, set $\gamma_k = \sigma + \epsilon\lambda_k$, then the argument of the square root above is exactly

$(1+\tau+(\delta+\gamma_k)\lambda_k)^2-4\gamma_k\lambda_k(\tau+\delta\lambda_k) = 1+2[\tau+(\delta+\gamma_k)\lambda_k]+[\tau+(\delta-\gamma_k)\lambda_k]^2 > 0.$

Moreover, the eigenfunctions of the operator $\mathbb{A}$ are $\hat{U}_k^{\pm} = (e_k, -\hat{\eta}_k^{\pm}e_k)$, where

$$\hat{\eta}_k^{\pm} = \frac{1}{2\gamma_k}\left(1+\tau+(\delta-\gamma_k)\lambda_k \pm \sqrt{(1+\tau+(\delta+\gamma_k)\lambda_k)^2-4\gamma_k\lambda_k(\tau+\delta\lambda_k)}\right),$$

$\{\lambda_k\}$ and $\{e_k\}$ are as given in (1.5.11)-(1.5.13).

Let $c_1 > 0$. There exists n such that $\lambda_n \geq 1$ and

$$\lambda_{n+1} - \lambda_n > \max\{4c_1(\tau+1), 4c_1\delta\}, \tag{4.3.3}$$

(cf. (2.4.5)). This implies that

$$\lambda_{n+1}^2 - \lambda_n^2 > \max\{4c_1(\tau+1), 4c_1\delta\}(\lambda_n + \lambda_{n+1}).$$

Denote

$$D_k = \gamma_k^2\lambda_k^2 + 2\gamma_k(1-\tau-\delta\lambda_k)\lambda_k + (1+\tau+\delta\lambda_k)^2,$$

$$\Delta_k = \frac{\gamma_k\lambda_{k+1}\sqrt{D_{k+1}}}{\tau+\delta\lambda_{k+1}} - \frac{\gamma_{k+1}\lambda_k\sqrt{D_k}}{\tau+\delta\lambda_k}, \qquad f_k = \frac{\lambda_{k+1}^2}{\tau+\delta\lambda_{k+1}} - \frac{\lambda_k^2}{\tau+\delta\lambda_k} - 2c_1,$$

$$h_k^{\pm} = \frac{\lambda_{k+1}^4}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_k^4}{(\tau+\delta\lambda_k)^2}, \quad a_k^{\pm} = \frac{\lambda_{k+1}(1+\tau+\delta\lambda_{k+1})}{\tau+\delta\lambda_{k+1}} \pm \frac{\lambda_k(1+\tau+\delta\lambda_k)}{\tau+\delta\lambda_k},$$

$$\alpha_k^{\pm} = \frac{\lambda_n\lambda_{k+1}(1+\tau+\delta\lambda_{k+1})}{\tau+\delta\lambda_{k+1}} \pm \frac{\lambda_n\lambda_{k+1}(1+\tau+\delta\lambda_k)}{\tau+\delta\lambda_k},$$

$$b_k^{\pm} = \frac{\lambda_{k+1}^3(1-\tau-\delta\lambda_{k+1})}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_k^3(1-\tau-\delta\lambda_k)}{(\tau+\delta\lambda_k)^2},$$

$$\beta_k^{\pm} = \frac{\lambda_n\lambda_{k+1}^3(1-\tau-\delta\lambda_{k+1})}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_{n+1}\lambda_k^3(1-\tau-\delta\lambda_k)}{(\tau+\delta\lambda_k)^2},$$

$$c_k^{\pm} = \frac{\lambda_{k+1}^2(1+\tau+\delta\lambda_{k+1})^2}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_k^2(1+\tau+\delta\lambda_k)^2}{(\tau+\delta\lambda_k)^2}, \qquad j_k^{\pm} = \sigma^2 c_k^{\pm} + 2\epsilon\sigma\omega_k^{\pm} + \epsilon^2\xi_k^{\pm},$$

$$\xi_k^{\pm} = \frac{\lambda_k^2\lambda_{k+1}^2(1+\tau+\delta\lambda_{k+1})^2}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_{k+1}^2\lambda_k^2(1+\tau+\delta\lambda_k)^2}{(\tau+\delta\lambda_k)^2}, \quad g_k^{\pm} = \sigma a_k^{\pm} + \epsilon\alpha_k^{\pm},$$

$$\omega_k^{\pm} = \frac{\lambda_k\lambda_{k+1}^2(1+\tau+\delta\lambda_{k+1})^2}{(\tau+\delta\lambda_{k+1})^2} \pm \frac{\lambda_{k+1}\lambda_k^2(1+\tau+\delta\lambda_k)^2}{(\tau+\delta\lambda_k)^2}, \quad i_k^{\pm} = \sigma b_k^{\pm} + \epsilon\beta_k^{\pm}.$$

We give the following result whose proof is by same method as in Lemma 2.1 and 3.2.

Lemma 4.3 *Provided that n is large enough for (4.3.3) to hold, there exists $\epsilon(n) > 0$ and $\sigma(n) > 0$ suitably small such that the following inequalities are satisfied:*

(i) $f_n \geq 0$ and $g_n \geq 0$.

(ii) $\Delta_n \geq 0$, *for every* $\epsilon \in (0, \epsilon(n)]$ and $\sigma \in (0, \sigma(n)]$.

(iii) $f_n^4 - 2f_n^2 h_n^+ + (h_n^-)^2 > 0$.

(iv) $b_n^- a_n^+ + f_n (a_n^-)^2 - (b_n^+ a_n^- + c_n^+ f_n) < 0.$

(v) $\gamma_n^2 \gamma_{n+1}^2 h_n^+ + 2\gamma_n \gamma_{n+1} i_n^+ + j_n^+ - (g_n^- + \gamma_n \gamma_{n+1} f_n)^2 > 0$, *for every* $\epsilon \in (0, \epsilon(n)]$ and $\sigma \in (0, \sigma(n)]$.

We have the following result.

Proposition 4.1 *For every $c_1 > 0$, there exists n (independent of ϵ), $\epsilon(n) > 0$, and $\sigma(n) > 0$ such that, for every $\epsilon \in (0, \epsilon(n)]$ and $\sigma \in (0, \sigma(n)]$, the spectral gap condition hold:*

$$\eta_{n+1}^- - \eta_n^- > c_1. \tag{4.3.4}$$

Proof. We have

$$\eta_{n+1}^- - \eta_n^- = \frac{1}{2\gamma_n \gamma_{n+1}} (g_n^- + \gamma_n \gamma_{n+1} f_n - \Delta_n) + c_1.$$

We proceeding like in the proof of Proposition 2.3. We can compute that $j_n^- = g_n^+ g_n^-$, and $(j_n^-)^2 + (g_n^-)^4 - 2j_n^+ (g_n^-)^2 = 0$. Thanks to Lemma 4.3, the inequality

$$\gamma_n \gamma_{n+1} f_n + g_n^- - \Delta_n > 0$$

is equivalent to

$$\begin{aligned}
&\gamma_n^3 \gamma_{n+1}^3 \left[(h_n^-)^2 + f_n^4 - 2f_n^2 h_n^+ \right] \\
&+ 4\gamma_n^2 \gamma_{n+1}^2 \left[h_n^- i_n^- + f_n^3 g_n^- - (h_n^+ f_n g_n^- + i_n^+ f_n^2) \right] \\
&+ 2\gamma_n \gamma_{n+1} \left[2(i_n^-)^2 + h_n^- j_n^- + 3f_n^2 (g_n^-)^2 - (h_n^+ (g_n^-)^2 + 4i_n^+ f_n g_n^- + j_n^+ f_n^2) \right] \\
&+ 4g_n^- \left[i_n^- g_n^+ + f_n (g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n) \right] < 0.
\end{aligned} \tag{4.3.5}$$

Denote

$$\ell_2 = h_n^- \beta_n^- + f_n^3 \alpha_n^- - (h_n^+ f_n \alpha_n^- + \beta_n^+ f_n^2),$$

$$\begin{aligned}\ell_1 =& 2\big(\epsilon(\beta_n^-)^2 + 2\sigma b_n^- \beta_n^-\big) + h_n^-\big(\epsilon\xi_n^- + 2\sigma\omega_n^-\big) + 3f_n^2\big(\epsilon(\alpha_n^-)^2 + 2\sigma a_n^- \alpha_n^-\big)\\ &- h_n^+\big(\epsilon(\alpha_n^-)^2 + 2\sigma a_n^- \alpha_n^-\big) - 4f_n\big(\sigma b_n^+ \alpha_n^- + \sigma\beta_n^+ a_n^- + \epsilon\beta_n^+ \alpha_n^+\big)\\ &- f_n^2\big(\epsilon\xi_n^+ + 2\sigma\omega_n^+\big),\end{aligned}$$

$$\begin{aligned}\ell_0 =& \alpha_n^-\big(i_n^- g_n^+ + f_n(g_n^-)^2 - (i_n^+ g_n^- + j_n^+ f_n)\big) + \sigma a_n^-\big(\sigma b_n^- \alpha_n^+ + \sigma\beta_n^- a_n^+ + \epsilon\beta_n^- \alpha_n^+\big)\\ &+ \sigma a_n^- f_n\bigg(\big(\epsilon(\alpha_n^-)^2 + 2\sigma a_n^- \alpha_n^-\big) - \big(\sigma b_n^+ \alpha_n^- + \sigma\beta_n^+ a_n^- + \epsilon\beta_n^+ \alpha_n^-\big)\bigg)\\ &+ \sigma a_n^- f_n\big(\epsilon\xi_n^+ + 2\sigma\omega_n^+\big).\end{aligned}$$

The inequality (4.3.5) is exactly

$$\begin{aligned}&\epsilon q_3[(h_n^-)^2 + f_n^4 - 2f_n^2 h_n^+] + 4\epsilon^2 q_2 \ell_2 + 2\epsilon^2 q_1 \ell_1 + 4\epsilon\ell_0\\ &+ \sigma^6[(h_n^-)^2 + f_n^4 - 2f_n^2 h_n^+] + 4\sigma^5\left[h_n^- b_n^- + f_n^3 a_n^- - (h_n^+ f_n a_n^- + b_n^+ f_n^2)\right]\\ &+ 2\sigma^4\left[2(b_n^-)^2 + h_n^- c_n^- + 3f_n^2(a_n^-)^2 - (h_n^+(a_n^-)^2 + 4b_n^+ f_n a_n^- + c_n^+ f_n^2)\right]\\ &+ 4\sigma^3 a_n^-\left[b_n^- a_n^+ + f_n(a_n^-)^2 - (b_n^+ a_n^- + c_n^+ f_n)\right] < 0.\end{aligned} \tag{4.3.6}$$

Note that on account of (iv) of Lemma 4.3, the inequality

$$\begin{aligned}&\sigma^6[(h_n^-)^2 + f_n^4 - 2f_n^2 h_n^+] + 4\sigma^5\left[h_n^- b_n^- + f_n^3 a_n^- - (h_n^+ f_n a_n^- + b_n^+ f_n^2)\right]\\ &+ 2\sigma^4\left[2(b_n^-)^2 + h_n^- c_n^- + 3f_n^2(a_n^-)^2 - (h_n^+(a_n^-)^2 + 4b_n^+ f_n a_n^- + c_n^+ f_n^2)\right]\\ &+ 4\sigma^3 a_n^-\left[b_n^- a_n^+ + f_n(a_n^-)^2 - (b_n^+ a_n^- + c_n^+ f_n)\right] < 0\end{aligned}$$

holds true for every $\sigma \in (0, \sigma(n)]$, for some $\sigma(n) > 0$.

Thus, the inequality (4.3.6) holds for every $\epsilon \in (0, \epsilon(n)]$, for some $\epsilon(n) > 0$. Finally, it follows from (4.3.6), that (4.3.4) holds for every $\epsilon \in (0, \epsilon(n)]$ and for every $\sigma \in (0, \sigma(n)]$. ∎

We now prove the following result.

Theorem 4.4 *Let the assumptions of Theorem 2.4 hold. Then, there exists $\epsilon(n)$ and $\sigma(n)$ such that, for every $\epsilon \in (0, \epsilon(n)]$ and for every $\sigma \in (0, \sigma(n)]$, System* (4.3.2) *has an inertial manifold $\widetilde{\mathfrak{M}}^{\alpha,\rho}_{\epsilon,\sigma}$ (with dimension independent of ϵ) in $\widetilde{K}_{\alpha,\rho} \cap \mathcal{W}_d$.*

Proof. Let $X_1, Y_1, X_{n1}, X_{n2}, \sigma_1$ and σ_2 be as in Section 2.4, however adapted to $\hat{U}_k^{\pm}$.

We introduce the scalar product $\langle\langle ., . \rangle\rangle$ in $\mathcal{W}_1$ defined by

$$\langle\langle U, V \rangle\rangle = \hat{\Psi}_1(\mathrm{P}_{X_n} U, \mathrm{P}_{X_n} V) + \hat{\Psi}_2(\mathrm{P}_{Y_n} U, \mathrm{P}_{Y_n} V), \tag{4.3.7}$$

for any $U, V \in \mathcal{U}_1$, where P_{X_n} and P_{Y_n} are, respectively, the projections from $\mathcal{W}_1$ onto X_n and Y_n and the functions $\hat{\Psi}_1 : X_n \times X_n \to \mathbb{R}$ and $\hat{\Psi}_2 : Y_n \times Y_n \to \mathbb{R}$ are defined by

$$\begin{aligned}
\hat{\Psi}_1(\mathscr{U}, \mathscr{V}) =& (1+\tau)(u,y) + (\delta+\sigma)(\nabla u, \nabla y) + ((\sigma I + \epsilon N)^{1/2} y, (\sigma I + \epsilon N)^{1/2} v) \\
& - \epsilon(Nu, Ny) + ((\sigma I + \epsilon N)^{1/2} u, (\sigma I + \epsilon N)^{1/2} z) \\
& + 2\sigma((\sigma I + \epsilon N)^{1/2} v, (\sigma I + \epsilon N)^{1/2} z), \\
\hat{\Psi}_2(\mathscr{U}, \mathscr{V}) =& (1+\tau)(u,y) + (\delta+\sigma)(\nabla u, \nabla y) + ((\sigma I + \epsilon N)^{1/2} y, (\sigma I + \epsilon N)^{1/2} v) \\
& + ((\sigma I + \epsilon N)^{1/2} u, (\sigma I + \epsilon N)^{1/2} z) \\
& + 2\sigma((\sigma I + \epsilon N)^{1/2} v, (\sigma I + \epsilon N)^{1/2} z),
\end{aligned}$$

with $\mathscr{U} = (u, v)$, $\mathscr{V} = (y, z)$ in X_n (or in Y_n).

Using Young's inequality, we have

$$\begin{aligned}\hat{\Psi}_1(\mathscr{U}, \mathscr{U}) \geq &(1+\tau)\|u\|^2 + (\delta+\sigma)\|\nabla u\|^2 - \epsilon\lambda_n\|u\|^2 - \frac{1}{\sigma}(\sigma\|u\|^2 + \epsilon\|\nabla u\|^2) \\ &+ \sigma\|(\sigma I + \epsilon N)^{1/2} v\|^2 \\ \geq &(\tau - \epsilon\lambda_n)\|u\|^2 + (\sigma + \delta - \frac{\epsilon}{\sigma})\|\nabla u\|^2 + \sigma(\sigma\|v\|^2 + \epsilon\|\nabla v\|^2) \\ \geq &C_4(\|u\|_1^2 + \epsilon\|v\|_1^2 + \|v\|^2), \qquad (4.3.8)\end{aligned}$$

for every $\mathscr{U} \in X_n$, and

$$\begin{aligned}\hat{\Psi}_2(\mathscr{U}, \mathscr{U}) \geq &(1+\tau)\|u\|^2 + (\delta+\sigma)\|\nabla u\|^2 - \frac{1}{\sigma}(\sigma\|u\|^2 + \epsilon\|\nabla u\|^2) \\ &+ \sigma\|(\sigma I + \epsilon N)^{1/2} v\|^2 \\ \geq &\tau\|u\|^2 + (\sigma + \delta - \frac{\epsilon}{\sigma})\|\nabla u\|^2 + \sigma(\sigma\|v\|^2 + \epsilon\|\nabla v\|^2) \\ \geq &C_5(\|u\|_1^2 + \epsilon\|v\|_1^2 + \|v\|^2), \qquad (4.3.9)\end{aligned}$$

for every $\mathscr{U} \in Y_n$ and for every $\epsilon \in (0, \epsilon(n)]$.

Thus, for $\hat{U}_l^- \in X_{n1}$ and $\hat{U}_l^+ \in X_{n2}$, noting that $(e_l, e_k) = \delta_{lk}$,

$$\hat{\eta}_l^+ + \hat{\eta}_l^- = \frac{1}{\sigma + \epsilon\lambda_l}(1 + \tau + \delta\lambda_l - (\sigma + \epsilon\lambda_l)\lambda_l), \quad \hat{\eta}_l^+\hat{\eta}_l^- = -\frac{\lambda_l}{\sigma + \epsilon\lambda_l},$$

we have that

$$\begin{aligned}\langle\langle \hat{U}_l^-, \hat{U}_l^+ \rangle\rangle &= 1 + \tau + (\delta+\sigma)\lambda_l - \epsilon\lambda_l^2 - (\sigma + \epsilon\lambda_l)(\hat{\eta}_l^- + \hat{\eta}_l^+) + 2\sigma(\sigma + \epsilon\lambda_l)\hat{\eta}_l^-\hat{\eta}_l^+ \\ &= 0.\end{aligned}$$

As a consequence, X_{n1} is orthogonal to X_{n2} and to Y_n, and the decomposition $\widetilde{K}_{\alpha,\rho} = X_{n1} \oplus X_{n2} \oplus Y_n$ is orthogonal with respect to the scalar product $\langle\langle ., . \rangle\rangle$ and we set $\mathcal{W}_1^1 = X_{n1}$ and $(\mathcal{W}_1^1)^\perp = X_{n2} \oplus Y_n$. Let $\mathcal{P}$ and $\mathcal{Q}$ be the unique orthogonal projections onto $\mathcal{W}_1^1$ and $(\mathcal{W}_1^1)^\perp$. We now define the norm

$$|||U||| = \langle\langle U, U\rangle\rangle^{1/2}. \tag{4.3.10}$$

Remark 4.3.1 *From* (4.3.8) *and* (4.3.9)*, we can deduce that there exists* $c_1, c_2 > 0$ *independent of* ϵ *such that for every* $\epsilon \in (0, \epsilon(n)]$,

$$c_1 \|U\|_{\mathcal{W}_{1,\epsilon}} \le |||U||| \le c_2 \|U\|_{\mathcal{W}_{1,\epsilon}}, \quad \forall U \in \mathcal{W}_1. \tag{4.3.11}$$

On account of (2.4.22), (2.4.23) and (4.3.11), for any given $U = (\phi, u), V = (\varphi, \vartheta) \in \widetilde{K}_{\alpha,\rho} \cap \mathcal{W}_d$, we have

$$|||\Gamma\mathbb{G}(U)||| \le c\|\Lambda\mathfrak{g}(\phi)\| \le c, \tag{4.3.12}$$

$$\begin{aligned} |||\Gamma\mathbb{G}(U) - \Gamma\mathbb{G}(V)||| &\le c\|\Lambda(\mathfrak{g}(\phi) - \mathfrak{g}(\varphi))\| \\ &\le c|||\Gamma U - \Gamma V|||, \end{aligned} \tag{4.3.13}$$

for all $\epsilon \in (0, \epsilon(n)]$, where $c > 0$ is independent of ϵ. Thus, $\mathbb{G} : \mathcal{W}_d \to \mathcal{W}_d$ is globally Lipschitz continuous.

Moreover, there exist $C_1, C_2 > 0$, independent of ϵ, such that (cf. (2.4.27)-(2.4.28))

$$\|\mathcal{Q}e^{s\mathbb{A}\mathcal{Q}}\|_{\mathcal{L}(\mathcal{Q}\mathcal{U}_d)} \le C_1 e^{s\mu_{n+1}^-}, \quad s < 0,$$

$$\|\mathcal{P}e^{-s\mathbb{A}\mathcal{P}}\|_{\mathcal{L}(\mathcal{P}\mathcal{U}_d)} \le C_2 e^{-s\mu_n^-}, \quad s \le 0.$$

It follows from Theorem 1.3 (see also [71, Chap. 9, Theorem 2.1] and [68]), that the semigroup $\tilde{S}_\epsilon(t)$ generated by Equation (4.3.2) admits an inertial manifold $\widetilde{\mathfrak{M}}_{\epsilon,\sigma}^{\alpha;\rho}$ in $\widetilde{K}_{\alpha,\rho} \cap \mathcal{W}_d$ of dimension n independent of ϵ, with respect to the metric induced by the norm $|||\Gamma\cdot|||$. Therefore, there exists a Lipschitz mapping $\tilde{\Phi}_{\epsilon,\sigma}^{\alpha;\rho}$:

$\widetilde{K}_{\alpha,\rho} \cap \mathcal{PW}_d \to \mathcal{QW}_d$ such that

$$\widetilde{\mathfrak{M}}^{\alpha,\rho}_{\epsilon,\sigma} = \{\tilde{\mathrm{p}} + \tilde{\Phi}^{\alpha,\rho}_{\epsilon,\sigma}(\tilde{\mathrm{p}}),\ \ \tilde{\mathrm{p}} \in \widetilde{K}_{\alpha,\rho} \cap \mathcal{PW}_d\}.$$

$\blacksquare$

4.4 A robust family of exponential attractors

Theorem 4.5 *For every $\epsilon \in (0,1]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor $\mathcal{E}^{\alpha,\rho}_\epsilon$ (with dimensions independent of ϵ) in $\widetilde{K}_{\alpha,\rho}$. Furthermore, there exist $0 < \varrho_1 \le 1$ and $M_1 > 0$ (all independent of ϵ) such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{W}_{1,\epsilon}}(\mathcal{E}^{\alpha,\rho}_\epsilon, \mathcal{E}^{\alpha,\rho}_0) \le M_1 \epsilon^{\varrho_1}, \tag{4.4.1}$$

where $\mathcal{E}^{\alpha,\rho}_0$ is an exponential attractor for the semigroup $S_0(t)$ on $K_{\alpha,\rho}$.

Proof. Let $z_1, z_2 \in \mathcal{B}_2,\ z_1 = (\phi^1_0, u^1_0)$ and $z_2 = (\phi^2_0, u^2_0)$ be initial data for two solutions (ϕ_1, u_1) and (ϕ_2, u_2) of (4.0.1) respectively.

We set $(\phi(t), u(t)) = S_\epsilon(t)z_1 - S_\epsilon(t)z_2$, $\tilde{\phi}_0 = \phi^1_0 - \phi^2_0$, $\tilde{u}_0 = u^1_0 - u^2_0$. Furthermore, we perform the splitting

$$(\phi(t), u(t)) = (\chi(t), \vartheta(t)) + (\Psi(t), \upsilon(t)),$$

where $\quad K_\epsilon(z_1, z_2) = (\chi(t), \vartheta(t)) \quad \text{and} \quad L_\epsilon(z_1, z_2) = (\Psi(t), \upsilon(t)),$

satisfy respectively the problems

$$\begin{cases} \tau\chi_t + N(\delta\chi_t + N\chi + g(\phi_1) - g(\phi_2) - \vartheta) + \chi = \phi, \\ \sigma\vartheta_t + \epsilon N\vartheta_t + \chi_t + N\vartheta + \vartheta = u, \\ \chi|_{t=0} = 0, \quad \vartheta|_{t=0} = 0, \end{cases} \tag{4.4.2}$$

and

$$\begin{cases} \tau\Psi_t + N(\delta\Psi_t + N\Psi - \upsilon) + \Psi = 0, \\ \sigma\upsilon_t + \epsilon N\upsilon_t + \Psi_t + N\upsilon + \upsilon = 0, \\ \Psi|_{t=0} = \tilde{\phi}_0, \quad \upsilon|_{t=0} = \tilde{u}_0. \end{cases} \tag{4.4.3}$$

Computations show that

$$\begin{aligned} & m(\Psi(t)) = m(\tilde{\phi}_0)e^{-\frac{1}{\tau}t}, \\ & m(\upsilon(t)) = m(\tilde{u}_0)e^{-\frac{1}{\sigma}t} + \frac{1}{\tau-\sigma}m(\tilde{\phi}_0)\big(e^{-\frac{1}{\tau}t} - e^{-\frac{1}{\sigma}t}\big), \\ & m(\chi(t)) = m(\tilde{\phi}_0)\big(1 - e^{-\frac{1}{\tau}t}\big), \\ & m(\vartheta(t)) = m(\tilde{u}_0)\big(1 - e^{-\frac{1}{\sigma}t}\big) - \frac{1}{\tau-\sigma}m(\tilde{\phi}_0)\big(e^{-\frac{1}{\tau}t} - e^{-\frac{1}{\sigma}t}\big). \end{aligned} \tag{4.4.4}$$

In this proof, $c > 0$ denotes a generic constant that is independent of ϵ but may depend on the size of $\mathcal{B}_1$ but not explicitly on the initial data.

Firstly, we multiply $(4.4.3)_1$ by $N^{-1}\bar{\Psi}_t$ and $(4.4.3)_2$ by $\bar{\upsilon}$, we integrate over Ω, and finally add the resulting equation to have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\big(\|\nabla\Psi\|^2 + \|\bar{\Psi}\|^2 + \epsilon\|\nabla\upsilon\|^2 + \sigma\|\bar{\upsilon}\|^2\big) + \tau\|N^{-1/2}\bar{\Psi}_t\|_{-1}^2 + \delta\|\bar{\Psi}_t\|^2 \\ & + \|\nabla\upsilon\|^2 + \|\bar{\upsilon}\|^2 = 0. \end{aligned} \tag{4.4.5}$$

Next, we multiply $(4.4.3)_1$ by $N^{-1}\bar{\Psi}$ and integrate over Ω, to get

$$\frac{1}{2}\frac{d}{dt}\left[\tau\|N^{-1/2}\bar{\Psi}\|^2 + \delta\|\bar{\Psi}\|^2\right] + \|\nabla\Psi\|^2 + \|N^{-1/2}\bar{\Psi}\|^2 - (\upsilon, \bar{\Psi}) = 0. \tag{4.4.6}$$

We sum (4.4.5) and κ(4.4.6), for $\kappa > 0$ to be appropriately chosen later, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_3(t) + \tau\|N^{-1/2}\bar{\Psi}_t\|^2 + \delta\|\bar{\Psi}_t\|^2 + \|\nabla\upsilon\|^2 + \|\bar{\upsilon}\|^2 + \kappa\|\nabla\Psi\|^2 \\ &+ \kappa\|N^{-1/2}\bar{\Psi}\|^2 - \kappa(w, \bar{\Psi}) = 0,\end{aligned} \tag{4.4.7}$$

where

$$E_3(t) = \|\nabla\Psi\|^2 + \epsilon\|\nabla\upsilon\|^2 + \sigma\|\bar{\upsilon}\|^2 + \kappa\tau\|N^{-1/2}\bar{\Psi}\|^2 + \kappa\delta\|\bar{\Psi}\|^2.$$

We observe that $-\kappa(\upsilon, \bar{\Psi}) \geq -\frac{\kappa}{2}\|\nabla\upsilon\|^2 - \frac{\kappa}{2}\|N^{-1/2}\bar{\Psi}\|^2$, and that there exists $c_1 > 0$ such that

$$\begin{aligned}c_1E_3(t) \leq& \tau\|N^{-1/2}\bar{\Psi}_t\|^2 + \delta\|\bar{\Psi}_t\|^2 + \left(1 - \frac{\kappa}{2}\right)\|\nabla\upsilon\|^2 + \|\bar{\upsilon}\|^2 + \kappa\|\nabla\Psi\|^2 \\ &+ \frac{\kappa}{2}\|N^{-1/2}\bar{\Psi}\|^2\end{aligned}$$

On account of the above last two inequalities, (4.4.7) gives

$$\frac{d}{dt}E_3(t) + 2c_1E_5(t) \leq 0, \tag{4.4.8}$$

then applying Gronwall's lemma on $(0, t)$, we obtain that

$$E_3(t) \leq E_3(0)e^{-2c_1t}, \quad \forall t \geq 0. \tag{4.4.9}$$

There exists $c_2 > 0$ such that for every $t \geq 0$,

$$\|(\Psi(t), \upsilon(t))\|^2_{\mathcal{W}_{1,\epsilon}} - |m(\Psi)|^2 - (\epsilon + \sigma)|m(\upsilon)|^2 \leq E_3(t) \leq c_1\|(\Psi(t), \upsilon(t))\|^2_{\mathcal{W}_{1,\epsilon}}.$$

From the above inequality, (4.4.9) together with $(4.4.4)_1$ and $(4.4.4)_2$, we deduce that

$$\|(\Psi(t), \upsilon(t))\|^2_{\mathcal{W}_{1,\epsilon}} \leq ce^{-\nu^2 t}\|(\tilde{\phi}_0, \tilde{u}_0)\|^2_{\mathcal{W}_{1,\epsilon}},$$

where $\nu^2 = \max\left\{2c_1, \frac{2}{\tau}, \frac{2}{\sigma}\right\}$. Hence,

$$\|L_\epsilon(z_1, z_2)\|_{\mathcal{W}_{1,\epsilon}} \leq ce^{-\nu t}\|z_1 - z_2\|_{\mathcal{W}_{1,\epsilon}}. \tag{4.4.10}$$

Secondly, we multiply $(4.4.2)_1$ by χ_t and $(4.4.2)_2$ by $N\vartheta$, we integrate over Ω, then adding the resulting equations, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|N\chi\|^2 + \|\chi\|^2 + \epsilon\|N\vartheta\|^2 + \sigma\|\nabla\vartheta\|^2\big) + \|N\vartheta\|^2 + \|\nabla\vartheta\|^2 + \tau\|\chi_t\|^2 \\ + \delta\|\nabla\chi_t\|^2 = (\phi, \chi_t) + (u, N\vartheta) - (\nabla(g(\phi_1) - g(\phi_2)), \nabla\chi_t).\end{aligned} \tag{4.4.11}$$

Using Young's inequality we have

$$\begin{aligned}&|(\phi, \chi_t)| \leq \frac{\tau}{2}\|\chi_t\|^2 + \frac{1}{2\tau}\|\phi\|^2, \qquad |(u, N\vartheta)| \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|N\vartheta\|^2, \\ &|(\nabla(g(\phi_1) - g(\phi_2)), \nabla\chi_t)| \leq c\|\phi\|_1^2 + \frac{\delta}{2}\|\nabla\chi_t\|^2.\end{aligned}$$

Therefore, (4.4.11) yields

$$\begin{aligned}\frac{d}{dt}\big(\|N\chi\|^2 + \|\chi\|^2 + \epsilon\|N\vartheta\|^2 + \sigma\|\nabla\vartheta\|^2\big) + \|N\vartheta\|^2 + 2\|\nabla\vartheta\|^2 + \tau\|\chi_t\|^2 \\ + \delta\|\nabla\chi_t\|^2 \leq c(\|\phi\|_1^2 + \|\phi\|^2 + \|u\|^2),\end{aligned}$$

then using (4.1.7), we deduce that

$$\frac{d}{dt}\big(\|N\chi\|^2 + \|\chi\|^2 + \epsilon\|N\vartheta\|^2 + \sigma\|\nabla\vartheta\|^2\big) \leq ce^{c^\star t}\|(\tilde{\phi}_0, \tilde{u}_0)\|^2_{\mathcal{W}_{1,\epsilon}}. \tag{4.4.12}$$

Integrating (4.4.12) over (0,t) and exploiting $(4.4.4)_3$ and $(4.4.4)_4$ we get

$$\|\chi\|_2^2 + \epsilon\|\vartheta\|_2^2 + \sigma\|\vartheta\|_1^2 \leq ce^{c^\star t}\|(\tilde{\phi}_0, \tilde{u}_0)\|^2_{\mathcal{W}_{1,\epsilon}} + |m(\chi)|^2 + (\epsilon + \sigma)|m(\vartheta)|^2,$$

and it follows that $\|(\chi(t), \vartheta(t))\|_{\mathcal{W}_{2,\epsilon}} \leq ce^{c^\star t}\|(\tilde{\phi}_0, \tilde{u}_0)\|_{\mathcal{W}_{1,\epsilon}}$. Hence

$$\|K_\epsilon(z_1, z_2)\|_{\mathcal{W}_{2,\epsilon}} \leq ce^{c't}\|z_1 - z_2\|_{\mathcal{W}_{1,\epsilon}}. \tag{4.4.13}$$

Next, let $t \in [t^*, 2t^*]$. We set

$$(\phi(t), u(t)) = S_\epsilon(t)z_{01} - S_\epsilon(t)z_{02} = (\phi_1(t), u_1(t)) - (\phi_2(t), u_2(t))$$

Therefore, the pair $(\phi(t), u(t))$ satisfies

.

$$\begin{cases} \tau\phi_t + N(\delta\phi_t + N\phi + g(\phi_1) - g(\phi_2) - u) = 0, \\ u_t + \epsilon N u_t + \phi_t + Nu = 0, \\ \phi|_{t=0} = \phi_{01} - \phi_{02}, \quad u|_{t=0} = u_{01} - u_{02}. \end{cases} \tag{4.4.14}$$

We have that,

$$\begin{aligned} &\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{W}_{1,\epsilon}} \\ &\leq \|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{W}_{1,\epsilon}} + \|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{W}_{1,\epsilon}}, \ \forall t, t' \in [t^*, 2t^*]. \end{aligned}$$

Then, proceeding like in (2.5.45)-(2.5.46), we obtain

$$\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{W}_{1,\epsilon}} \leq c(\epsilon, t^*)|t' - t|^{1/2},$$

and that from (4.1.7), we have

$$\|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{W}_{1,\epsilon}} \leq c(t^*)\|z_{01} - z_{02}\|_{\mathcal{W}_{1,\epsilon}}, \ \forall t' > 0. \tag{4.4.15}$$

Hence, we conclude with

$$\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{W}_{1,\epsilon}} \leq c(\epsilon, t^*)(|t' - t|^{1/2} + \|z_{01} - z_{02}\|_{\mathcal{W}_{1,\epsilon}}). \tag{4.4.16}$$

Now, we check the assumptions of Theorem 1.2. Assumption 2 follows from estimates (4.4.10) and (4.4.13). Assumption 4 and 5 follow from (4.1.5) and

(4.4.16) respectively.

There exist $c, c' > 0$ (independent of ϵ) such that for any $(\phi_0, u_0) \in \mathcal{B}_2$, and any $\epsilon \in (0,1]$,

$$\|S_\epsilon(t)(\phi_0, u_0) - S_0(t)(\phi_0, u_0)\|^2_{\mathcal{U}_{1,\epsilon}} \leq c\epsilon e^{c't}, \ \ \forall t \geq 0, \tag{4.4.17}$$

(cf. Proof of Proposition 2.6). Hence Assumption 3 follows immediately from (4.4.17). Thus the existence of exponential attractors $\mathcal{E}^{\alpha,\rho}_\epsilon$ in $\mathcal{W}_1$ satisfying (4.4.1) according to Theorem 1.2. ▮

4.5 Continuity of inertial manifolds

We now want to prove some stability properties of the inertial manifolds. According to Theorem 2.4, the semigroup $S_0(t)$ for the unperturbed problem (2.4.4) possesses an inertial manifold $\mathfrak{M}^{\alpha,\rho}_{0,\sigma}$ on $K_{\alpha,\rho}$,

$$\mathfrak{M}^{\alpha,\rho}_{0,\sigma} = \{\bar{\mathrm{p}} + \Phi^{\alpha,\rho}(\bar{\mathrm{p}}), \ \ \bar{\mathrm{p}} \in K_{\alpha,\rho} \cap \mathcal{P}U_d\}.$$

For any arbitrary $R > 0$, we set

$$\widetilde{\mathfrak{M}}^{\alpha,\rho}_{\epsilon,\sigma,R} = \{(\phi, u), \ \ (\phi, u) = \tilde{\mathrm{p}} + \Phi^{\alpha,\rho}_{\epsilon,\sigma}(\tilde{\mathrm{p}}), \ \ |||\tilde{\mathrm{p}}||| \leq R\},$$

$$\mathfrak{M}^{\alpha,\rho}_{0,\sigma,R} = \{(\phi, u), \ \ (\phi, u) = \bar{\mathrm{p}} + \Phi^{\alpha,\rho}(\bar{\mathrm{p}}), \ \ |||\bar{\mathrm{p}}||| \leq R\}.$$

We prove the following result.

Theorem 4.6 *Let the assumptions of Theorem 4.4 hold. Then, there exists $M_3 > 0$ and $M_4 > 0$ (independent of ϵ) such that,*

$$\mathrm{dist}_{\mathcal{W}_{d,\epsilon}}(\mathfrak{M}^{\alpha,\rho}_{0,\sigma,R}, \widetilde{\mathfrak{M}}^{\alpha,\rho}_{\epsilon,\sigma}) \leq M_3\sqrt{\epsilon}, \tag{4.5.1}$$

$$\mathrm{dist}_{\mathcal{W}_{d,\epsilon}}(\widetilde{\mathfrak{M}}^{\alpha,\rho}_{\epsilon,\sigma,R}, \mathfrak{M}^{\alpha,\rho}_{0,\sigma}) \leq M_4\sqrt{\epsilon}, \quad \forall \epsilon \in (0, \tilde{\epsilon}(n)]. \tag{4.5.2}$$

Proof. Firstly, from Remark 4.3.1 we can deduce that, there exist c_1 and c_2 (c_2 independent of ϵ) such that

$$c_1\|U\|_{\mathcal{W}_{d,\epsilon}} \leq \|\|\Gamma U\|\| \leq c_2\|U\|_{\mathcal{W}_{d,\epsilon}}, \quad \forall U \in \mathcal{W}_d. \tag{4.5.3}$$

Let $U_0 = (\phi_0, u_0) \in \mathfrak{M}^{\alpha,\rho}_{0,\sigma,R}$, then there exists a complete trajectory $U(t) = (\phi(t), u(t))_{t\in\mathbb{R}}$ lying in $\mathfrak{M}^{\alpha,\rho}_{0,\sigma}$ and satisfies the non autonomous initial value problem:

$$\begin{cases} U_t + \mathbb{A}U + \mathbb{G}(U) = \mathbb{F}(t), \\ U(0) = (\phi_0, u_0), \end{cases} \tag{4.5.4}$$

where

$$\mathbb{F}(t) = \big(0, \epsilon(\sigma I + \epsilon N)^{-1} N u_t(t)\big).$$

We have that, there exists $M = M(n) > 0$, independent of ϵ, such that

$$\|\Gamma \nabla u_t(s)\| \leq M e^{-\zeta_n^- s}, \ \ \forall s \leq 0, \tag{4.5.5}$$

(cf. proof of (2.6.3)). Also, for any $c_1 > 0$ and for every $\sigma \in (0, \tilde{\sigma}(n)]$ and $\epsilon \in (0, \tilde{\epsilon}(n)]$, there holds

$$\begin{cases} \eta_n^- - \zeta_n + c_1 < 0, \\ \eta_{n+1}^- - \zeta_n - c_1 > 0. \end{cases} \tag{4.5.6}$$

where

$$\zeta_n = \frac{\mu_n^- + \mu_{n+1}^-}{2},$$

μ_n^- is the eigenvalue of the operator $\mathcal{A}$ (but setting $\varepsilon = \sigma$) given in Section 2.4. Note that (4.5.6) indeed holds by continuity property, since $\lim_{\epsilon \to 0^+} \eta_{n+1}^- - \zeta_n - c_1 = \mu_{n+1}^- - \mu_n^- - c_1 > 0$ and $\lim_{\epsilon \to 0^+} \eta_n^- - \zeta_n - c_1 = \mu_n^- - \mu_{n+1}^- - c_1 < 0$, for every $\sigma \in (0, \sigma(n)]$.

Let $(U_\epsilon(t))_{t \in \mathbb{R}}$ be a complete trajectory lying in $\widetilde{\mathfrak{M}}_{\epsilon,\sigma}^{\alpha,\rho}$ and solution to (4.3.2). The function $\mathrm{z}(t) = U(t) - U_\epsilon(t)$ is defined for all time $t \in \mathbb{R}$ and satisfies

$$\begin{cases} \mathrm{z}_t + \mathbb{A}\mathrm{z} + \mathbb{G}(U) - \mathbb{G}(U_\epsilon) = \mathbb{F}(t), \\ \mathrm{z}(0) = U_0 - U_\epsilon(0). \end{cases} \tag{4.5.7}$$

We write $\mathrm{z}(t) = \mathrm{p}(t) + \mathrm{q}(t)$, where $\mathrm{p} = \mathcal{P}\mathrm{z}$ and $\mathrm{q} = \mathcal{Q}\mathrm{z}$; and we have that

$$\begin{aligned} \mathrm{z}(t) =& e^{-\mathbb{A}\mathcal{P}t}\mathrm{p}(0) + \int_0^t e^{-\mathbb{A}\mathcal{P}(t-s)}\mathcal{P}\left[\mathbb{G}(U_\epsilon(s)) - \mathbb{G}(U(s)) + \mathbb{F}(s)\right] ds \\ &+ \int_{-\infty}^t e^{-\mathbb{A}\mathcal{Q}(t-s)}\mathcal{Q}\left[\mathbb{G}(U_\epsilon(s)) - \mathbb{G}(U(s)) + \mathbb{F}(s)\right] ds. \end{aligned} \tag{4.5.8}$$

Now, like in Section 2.4, we can choose $U_\epsilon(0)$ such that $\mathrm{p}(0) = 0$. Also, we have that $\mathrm{z} \in \mathcal{C}_{\zeta_n}((-\infty, 0]; \mathcal{W}_d)$, due to (4.5.6). Thus,

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \\
=&\sup_{t\le 0} e^{t\zeta_n}\left|\!\left|\!\left|\left\{\int_0^t e^{\mathbb{A}\mathcal{P}(s-t)}\mathcal{P}[\Gamma\mathbb{G}(U(s))-\Gamma\mathbb{G}(U_\epsilon(s))+\Gamma\mathbb{F}(s)]ds\right.\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left.+\int_{-\infty}^t e^{\mathbb{A}\mathcal{Q}(s-t)}\mathcal{Q}[\Gamma\mathbb{G}(U(s))-\Gamma\mathbb{G}(U_\epsilon(s))+\Gamma\mathbb{F}(s)]ds\right\}\right|\!\right|\!\right| \\
\le& c_3\sup_{t\le 0}\left\{\int_t^0 e^{(\eta_n^- -\zeta_n)(s-t)}ds+\int_{-\infty}^t e^{(\eta_{n+1}^- -\zeta_n)(s-t)}ds\right\}\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \\
&+c_2\sup_{t\le 0}\left\{e^{t\zeta_n}\int_t^0 e^{(s-t)\eta_n^-}|\!|\!|\Gamma\mathbb{F}(s)|\!|\!|ds+e^{t\zeta_n}\int_{-\infty}^t e^{(s-t)\eta_{n+1}^-}|\!|\!|\Gamma\mathbb{F}(s)|\!|\!|ds\right\} \\
\le& c_3\sup_{t\le 0}\left\{e^{-(\eta_n^- -\zeta_n)t}\int_t^0 e^{(\eta_n^- -\zeta_n)s}ds+e^{-(\eta_{n+1}^- -\zeta_n)t}\int_{-\infty}^t e^{(\eta_{n+1}^- -\zeta_n)s}ds\right\}\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \\
&+c_2\sup_{t\le 0}\left\{e^{(\zeta_n-\eta_n^-)t}\int_t^0 e^{s\eta_n^-}\|(\sigma I+\epsilon N)^{1/2}\Gamma\big(\epsilon(\sigma I+\epsilon N)^{-1}Nu_t(s)\big)\|ds\right. \\
&\qquad\qquad \left.+e^{(\zeta_n-\eta_{n+1}^-)t}\int_{-\infty}^t e^{s\eta_{n+1}^-}\|(\sigma I+\epsilon N)^{1/2}\Gamma\big(\epsilon(\sigma I+\epsilon N)^{-1}Nu_t(s)\big)\|ds\right\}.
\end{aligned}
\tag{4.5.9}
$$

We have that

$$
\begin{aligned}
|\!|\!|\Gamma\mathbb{F}(t)|\!|\!|^2 &= \epsilon\|\sqrt{\epsilon}(\sigma I+\epsilon N)^{-1/2}N^{1/2}\Gamma N^{1/2}u_t(t)\|^2 \\
&= \epsilon\|\Gamma\nabla u_t(t)\|^2.
\end{aligned}
\tag{4.5.10}
$$

We infer from (4.5.5), (4.5.9) and (4.5.10) that

$$
\begin{aligned}
&\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \\
&\le c_3\sup_{t\le 0}\left\{\frac{1}{\zeta_n-\eta_n^-}\big[1-e^{-(\eta_n^- -\zeta_n)t}\big]+\frac{1}{\eta_{n+1}^- -\zeta_n}\right\}\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \\
&\quad +c_2\sqrt{\epsilon}\sup_{t\le 0}\left\{e^{(\zeta_n-\eta_n^-)t}\int_t^0 e^{(\eta_n^- -\zeta_n)s}ds+e^{(\zeta_n-\eta_{n+1}^-)t}\int_{-\infty}^t e^{(\eta_{n+1}^- -\zeta_n)s}ds\right\} \\
&\le c_3\left(\frac{1}{\zeta_n-\eta_n^-}+\frac{1}{\eta_{n+1}^- -\zeta_n}\right)\|\mathrm{z}\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)}+c_2\sqrt{\epsilon}\left(\frac{1}{\zeta_n-\eta_n^-}+\frac{1}{\eta_{n+1}^- -\zeta_n}\right).
\end{aligned}
\tag{4.5.11}
$$

We then deduce from (4.5.11), with an appropriate choice of n, that

$$\|z\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} \leq \frac{1}{2}\|z\|_{\mathcal{C}_{\zeta_n}((-\infty,0];\mathcal{W}_d)} + M\sqrt{\epsilon}. \tag{4.5.12}$$

Hence we obtain (cf. (2.6.9) through (2.6.10))

$$\|\|\Gamma U_0 - \Gamma U_\epsilon(0)\|\| \leq M\sqrt{\epsilon}. \tag{4.5.13}$$

Estimates (4.5.3) and (4.5.13) imply the lower semicontinuity estimate (4.5.1).

Finally, the proof of (4.5.2) is similar to that of (2.6.18). ∎

CHAPTER 5

A PARABOLIC-HYPERBOLIC PHASE-FIELD SYSTEM

We consider the following parabolic-hyperbolic phase-field system:

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi) - u = 0, \\ u_t + \phi_t - \Delta u = 0, \\ \partial_n \phi|_{\partial\Omega} = u|_{\partial\Omega} = 0, \\ \phi(0) = \phi_0, \phi_t(0) = \phi_1, u(0) = u_0, \end{cases} \tag{5.0.1}$$

where $\epsilon \in (0, 1]$ and g satisfies assumptions (1.5.3)-(1.5.6) with $p \in [0, 1]$ when $d = 3$.

We shall construct a robust family of exponential attractors which are both upper and lower semicontinuous at $\epsilon = 0$.

We define the Hilbert space $\mathcal{H}_r = V_r \times V_{r-1} \times H_0^{r-1}, \quad r \geq 1$, endowed with the norm

$$\|(\varphi, \psi, v)\|_{\mathcal{H}_{r,\epsilon}} = (\|\varphi\|_r^2 + \epsilon\|\psi\|_{r-1}^2 + \|v\|_{r-1}^2)^{1/2},$$

where we understand that $V_0 = H^0(\Omega) = L^2(\Omega)$. Hence, we denote $\mathcal{H}_{1,0} = H^1(\Omega) \times L^2(\Omega)$, endowed with the norm $\|(\cdot, \cdot)\|_{\mathcal{H}_{1,0}} = (\| \cdot \|_1^2 + \| \cdot \|^2)^{1/2}$.

5.1 A priori estimates

We multiply $(5.0.1)_1$ by ϕ_t and $(5.0.1)_2$ by u, then integrate over Ω. Adding the resulting equations, we obtain

$$\frac{d}{dt}E_1(t) + 2\|\phi_t\|^2 + 2\|\nabla u\|^2 = 0. \tag{5.1.1}$$

where

$$E_1(t) = \|\nabla\phi\|^2 + \|\phi\|^2 + \epsilon\|\phi_t\|^2 + \|u\|^2 + 2\int_\Omega G(\phi)dx.$$

From (1.5.3) as well as (2.1.10), we deduce that

$$\|(\phi, \phi_t, u)\|_{\mathcal{H}_{1,\epsilon}}^2 - \alpha_1 \leq E_1(t) \leq \alpha_2(\|\phi\|_1^{p+3} + \epsilon\|\phi_t\|^2 + \|u\|^2 + 1), \tag{5.1.2}$$

for some $\alpha_1, \alpha_2 > 0$ independent of ϵ. Thus integrating (5.1.1) over $(0, t)$, and on account of (5.1.2) we obtain that

$$\int_0^t (\|\phi_t(s)\|^2 + \|\nabla u(s)\|^2)ds \leq E_3(0) + \alpha_1, \quad \forall t \geq 0.$$

Hence by (5.1.2) again, we get

$$\int_0^\infty (\|\phi_t(s)\|^2 + \|\nabla u(s)\|^2)ds \leq c(\|\phi_0\|_1^{p+3} + \epsilon\|\phi_1\|^2 + \|u_0\|^2 + 1). \tag{5.1.3}$$

Let (ϕ^1, u^1) and (ϕ^2, u^2) be two solutions of (5.0.1). Set $\phi = \phi^1 - \phi^2$, $\phi_t = \phi_t^1 - \phi_t^2$ and $u = u^1 - u^2$, then $\phi(0) = 0$, $\phi_t(0) = 0$ and $u(0) = 0$. The pair (ϕ, ϕ_t, u)

satisfies the problem

$$
\begin{cases}
\epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi^1) - g(\phi^2) - u = 0, \\
u_t + \phi_t - \Delta u = 0, \\
\phi(0) = \phi_t(0) = u(0) = 0.
\end{cases}
\tag{5.1.4}
$$

Proceeding like in the proof of uniqueness in Theorem 2.1, we obtain

$$
\frac{d}{dt}\big(\|\nabla\phi\|^2 + \|\phi\|^2 + \epsilon\|\phi_t\|^2 + \|u\|^2\big) \le \widetilde{M}(t)\|\phi\|_1^2, \tag{5.1.5}
$$

where

$$
\widetilde{M}(t) = \begin{cases}
c \sup\limits_{\theta\in[0,1]} \|g'(\theta\phi_1 + (1-\theta)\phi_2)\|^2_{L^\infty(\Omega)}, & \text{if} \quad d = 1, \\
c(\|\phi^1\|_1^{2p+2} + \|\phi^2\|_1^{2p+2} + 1), & \text{if} \quad d = 2, 3.
\end{cases}
$$

Noting that $t \mapsto \widetilde{M}(t)$ is $L^1(0,T)$, and applying Gronwall's lemma to (5.1.5) on $(0,t)$, we deduce that

$$
\|(\phi(t), \phi_t(t), u(t))\|^2_{\mathcal{H}_{1,\epsilon}} \le e^{\int_0^t \widetilde{M}(s)ds}\|(\phi(0), \phi_t(0), u(0))\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0. \tag{5.1.6}
$$

We state a well-posedness result which proof can be found in [45, Theorem 3.4].

Theorem 5.1 *We assume that* (1.5.3)-(1.5.6) *hold. If* $(\phi_0, \phi_1, u_0) \in \mathcal{H}_1$, *then* (5.0.1) *possesses a unique solution* (ϕ, u) *such that*

$$
(\phi, \phi_t, u) \in \mathcal{C}([0,T]; \mathcal{H}_1)
$$

for any $T > 0$. *Moreover, if* $(\phi_0, \phi_1, u_0) \in \mathcal{H}_2$, *then* $(\phi, \phi_t, u) \in \mathcal{C}([0,T]; \mathcal{H}_2)$.

On account of Theorem 5.1 we can define the semigroup

$$S_\epsilon(t) : \mathcal{H}_1 \to \mathcal{H}_1, \quad (\phi_0, \phi_1, u_0) \mapsto (\phi(t), \phi_t(t), u(t)), \quad \forall t \geq 0,$$

where $(\phi(t), \phi_t(t), u(t))$ is the solution to problem (5.0.1) at time t. The semigroup $S_\epsilon(t)$ is strongly continuous [cf. (5.1.6)].

It is also known from [45] that the semigroup $S_\epsilon(t) : \mathcal{H}_j \to \mathcal{H}_j$ has bounded absorbing sets $\mathcal{B}_j$ in $\mathcal{H}_j$ of the form:

$$\mathcal{B}_j = \{(\varphi, \psi, v) \in \mathcal{H}_j, \|(\varphi, \psi, v)\|_{\mathcal{H}_{j,\epsilon}} \leq r_j\}, \quad j = 1, 2,$$

where $r_j > 0$ is independent of ϵ. In fact they are exponentially attracting sets.

5.2 Exponential attractors

We observe that the solution to the unperturbed problem for the pair (ϕ, u) at any time t is given by $(\phi(t), u(t)) = S(t)(\phi_0, u_0)$ and $\phi_t = \mathscr{L}(\phi(t), u(t))$, where

$$\mathscr{L}(\varphi, \vartheta) = -(N\varphi + \varphi - g(\varphi) - \vartheta). \tag{5.2.1}$$

Let $z_1, z_2 \in \mathcal{B}_2$, $z_1 = (\phi_0^1, \phi_1^1, u_0^1)$ and $z_2 = (\phi_0^2, \phi_1^2, u_0^2)$ be initial data for two solutions (ϕ^1, u^1) and (ϕ^2, u_2) of (5.0.1) respectively.
We set $(\phi(t), \phi_t(t), u(t)) = S_\epsilon(t)z_1 - S_\epsilon(t)z_2$, $\tilde{\phi}_0 = \phi_0^1 - \phi_0^2$, $\tilde{\phi}_1 = \phi_1^1 - \phi_1^2$, $\tilde{u}_0 = u_0^1 - u_0^2$. Furthermore, we perform the splitting

$$(\phi(t), \phi_t(t), u(t)) = (\chi(t), \chi_t(t), \vartheta(t)) + (\Psi(t), \Psi_t(t), \upsilon(t)),$$

where $K_\epsilon(z_1, z_2) = (\chi(t), \chi_t(t), \vartheta(t))$ and $L_\epsilon(z_1, z_2) = (\Psi(t), \Psi_t(t), \upsilon(t))$ satisfy respectively the problems:

$$\begin{cases} \epsilon\chi_{tt} + \chi_t - \Delta\chi_t + \chi + g(\phi_1) - g(\phi_2) - \vartheta = 0, \\ \vartheta_t + \chi_t - \Delta\vartheta = 0, \\ \chi|_{t=0} = 0, \quad \chi_t|_{t=0} = 0, \quad \vartheta|_{t=0} = 0, \end{cases} \tag{5.2.2}$$

and

$$\begin{cases} \epsilon\Psi_{tt} + \Psi_t - \Delta\Psi + \Psi - \upsilon = 0, \\ \upsilon_t + \Psi_t - \Delta\upsilon = 0, \\ \Psi|_{t=0} = \tilde{\phi}_0, \quad \Psi_t|_{t=0} = \tilde{\phi}_1, \quad \upsilon|_{t=0} = \tilde{u}_0. \end{cases} \tag{5.2.3}$$

Proposition 5.1 *There exist $c, c', c_7 > 0$ independent of ϵ such that*

$$\|L_\epsilon(z_1, z_2)\|_{\mathcal{H}_{1,\epsilon}} \leq ce^{-c_7 t}\|z_1 - z_2\|_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \geq 0, \tag{5.2.4}$$

$$\text{and} \qquad \|K_\epsilon(z_1, z_2)\|_{\mathcal{H}_{2,\epsilon}} \leq ce^{c' t}\|z_1 - z_2\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \geq 0. \tag{5.2.5}$$

Proof. Firstly, we multiply $(5.2.3)_1$ by Ψ_t and $(5.2.3)_2$ by υ, integrate over Ω, then addition of the resulting equations give

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\Psi\|^2 + \|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2\big) + \|\Psi_t\|^2 + \|\nabla\upsilon\|^2 = 0. \tag{5.2.6}$$

Next, we multiply $(5.2.3)_1$ by Ψ to obtain

$$\frac{1}{2}\frac{d}{dt}\big[\|\Psi\|^2 + 2\epsilon(\Psi, \Psi_t)\big] - \epsilon\|\Psi_t\|^2 + \|\nabla\Psi\|^2 + \|\Psi\|^2 - (\upsilon, \Psi) = 0,$$

then we deduce that

$$\frac{1}{2}\frac{d}{dt}\left[\|\Psi\|^2 + 2\epsilon(\Psi, \Psi_t)\right] + \|\nabla\Psi\|^2 + \frac{1}{2}\|\Psi\|^2 + 2\epsilon(\Psi_t, \Psi) \leq 5\epsilon\|\Psi_t\|^2 + c\|\nabla\upsilon\|^2. \tag{5.2.7}$$

Summing (5.2.6) and κ(5.2.7), for some $\kappa \in (0,1)$ small enough, we get

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\Psi\|^2 + (1+\kappa)\|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t)\big) + \kappa\|\nabla\Psi\|^2 + \frac{\kappa}{2}\|\Psi\|^2$$
$$+ \epsilon(1-5\kappa)\|\Psi_t\|^2 + (1-c\kappa)\|\nabla\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t) \leq 0.$$

Hence, there exists $c_7 > 0$ (independent of ϵ) such that

$$\frac{d}{dt}E_2(t) + c_7 E_2(t) \leq 0,$$

where $E_2(t) = \|\nabla\Psi\|^2 + (1+\kappa)\|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t)$. Simple integration over $(0,t)$ gives

$$E_2(t) \leq e^{-c_7 t}E_2(0), \quad \forall t \geq 0. \tag{5.2.8}$$

Clearly, by Young's inequality, there exist $b_3, b_4 > 0$ (independent of ϵ) such that

$$b_3\|(\Psi, \Psi_t, \upsilon)\|^2_{\mathcal{H}_{1,\epsilon}} \leq E_2(t) \leq b_4\|(\Psi, \Psi_t, \upsilon)\|^2_{\mathcal{H}_{1,\epsilon}}. \tag{5.2.9}$$

It follows from (5.2.8) and (5.2.9), that

$$\|(\Psi, \Psi_t, \upsilon)\|^2_{\mathcal{H}_{1,\epsilon}} \leq e^{-c_7 t}\|(\tilde{\phi}_0, \tilde{\phi}_1, \tilde{u}_0)\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \geq 0.$$

Hence (5.2.4).

Secondly, we multiply $(5.2.2)_1$ by χ_t and $(5.2.2)_2$ by ϑ, integrate over Ω, then addition of the resulting equations gives

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\chi\|^2 + \|\chi\|^2 + \epsilon\|\chi_t\|^2 + \|\vartheta\|^2\big) + \|\chi_t\|^2 + \|\nabla\vartheta\|^2 = -(g(\phi^1) - g(\phi^2), \chi_t).$$

We have that $\|g(\phi^1) - g(\phi^2)\| \leq \|g'(\theta\phi^1 + (1-\theta)\phi^2)\|_{L^\infty(\Omega)}\|\phi\|$, where $\theta \in [0,1]$. It follows that

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\chi\|^2+\|\chi\|^2+\epsilon\|\chi_t\|^2+\|\vartheta\|^2\big)+\frac{1}{2}\|\chi_t\|^2+\|\nabla\vartheta\|^2\le\|\phi\|^2. \tag{5.2.10}$$

Integrating (5.2.10) over $(0,t)$, then on account of (5.1.6), we deduce that

$$\|\chi\|_1^2+\epsilon\|\chi_t\|^2+\|\vartheta\|^2\le ce^{c't}\|(\tilde{\phi}_0,\tilde{\phi}_1,\tilde{u}_0)\|^2_{\mathcal{H}_{1,\epsilon}},\quad\forall t\ge 0. \tag{5.2.11}$$

Next, we multiply $(5.2.2)_1$ by $-\Delta\chi_t$ and $(5.2.2)_2$ by $N\vartheta$, integrate over Ω, then addition of the resulting equations give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\Delta\chi\|^2+\|\nabla\chi\|^2+\epsilon\|\nabla\chi_t\|^2+\|\nabla\vartheta\|^2\big)+\|\nabla\chi_t\|^2+\|\Delta\vartheta\|^2\\&=-(\nabla(g(\phi^1)-g(\phi^2)),\nabla\chi_t).\end{aligned}$$

We have that $\|\nabla(g(\phi^1)-g(\phi^2))\|\le c\|\phi\|_1$. It follows that

$$\frac{1}{2}\frac{d}{dt}\big(\|\Delta\chi\|^2+\|\nabla\chi\|^2+\epsilon\|\nabla\chi_t\|^2+\|\nabla\vartheta\|^2\big)+\frac{1}{2}\|\nabla\chi_t\|^2+\|\Delta\vartheta\|^2\le c\|\phi\|_1^2. \tag{5.2.12}$$

Integrating (5.2.12) over $(0,t)$, then, again on account of (5.1.6), we deduce that

$$\|\Delta\chi\|^2+\|\nabla\chi\|^2+\epsilon\|\nabla\chi_t\|^2+\|\nabla\vartheta\|^2\le ce^{c't}\|(\tilde{\phi}_0,\tilde{\phi}_1,\tilde{u}_0)\|^2_{\mathcal{H}_{1,\epsilon}},\quad\forall t\ge 0. \tag{5.2.13}$$

On account of (5.2.11) and (5.2.13), we obtain that

$$\|(\chi,\chi_t,\vartheta)\|^2_{\mathcal{H}_{2,\epsilon}}\le ce^{c't}\|(\tilde{\phi}_0,\tilde{\phi}_1,\tilde{u}_0)\|^2_{\mathcal{H}_{1,\epsilon}},\quad\forall t\ge 0.$$

Hence (5.2.5). ∎

We prove the following result.

Theorem 5.2 *For every $\epsilon\in(0,1]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor (with dimensions independent of ϵ) $\mathcal{E}_\epsilon$ in $\mathcal{H}_1$.*

Proof. Let $t\in[t^*,2t^*]$. We set $(\phi(t),\phi_t(t),u(u))=S_\epsilon(t)z_{01}-S_\epsilon(t)z_{02}=(\phi^1(t),\phi^1_t(t),u^1(t))-(\phi^2(t),\phi^2_t(t),u^2(t))$. Therefore, the triplet $(\phi(t),\phi_t(t),u(u))$

is solution to the problem

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi_1) - g(\phi_2) - u = 0, \\ u_t + \phi_t - \Delta u = 0, \\ \phi|_{t=0} = \phi^{01} - \phi^{02}, \quad \phi_t|_{t=0} = \phi_1^{01} - \phi_1^{02}, \quad u|_{t=0} = u^{01} - u^{02}. \end{cases} \tag{5.2.14}$$

On account of (5.1.6) we obtain

$$\|S_\epsilon(t)z_{01} - S_\epsilon(t)z_{02}\|_{\mathcal{H}_{1,\epsilon}} \leq c(t^*)\|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \quad t \geq 0, \tag{5.2.15}$$

where $c(t^*) > 0$ is independent of ϵ. Now, we multiply $(5.0.1)_1$ and $(5.0.1)_2$ by $-\Delta\phi$ and $-\Delta u$ respectively, integrate over Ω then add the resulting equations, to deduce that

$$\begin{aligned} &\frac{d}{dt}\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \epsilon\|\nabla\phi_t\|^2 + \|\nabla u\|^2\big) + \|\nabla\phi_t\|^2 + \|\Delta u\|^2 \\ &\leq \frac{1}{2}\|g'(\phi)\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2 \\ &\leq c\|\nabla\phi\|^2. \end{aligned}$$

Integrating over $(0,t)$ and recalling (5.1.2), we get

$$\begin{aligned} &\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \epsilon\|\nabla\phi_t\|^2 + \|\nabla u\|^2 + \int_0^t (\|\nabla\phi_t(s)\|^2 + \|\Delta u(s)\|^2)ds \\ &\leq c(t+1), \quad \forall t \geq 0. \end{aligned} \tag{5.2.16}$$

It then follows from (5.1.3) and (5.2.16) that

$$\int_0^t (\|\phi_t(s)\|_1^2 + \|\Delta u(s)\|^2)ds \leq c(t+1), \quad \forall t \geq 0. \tag{5.2.17}$$

Next, from $(5.0.1)_1$, we deduce that

$$\epsilon^2 \int_0^t \|\phi_{tt}(s)\|^2 ds \le \int_0^t (\|\phi_t(s)\|^2 + \|\Delta\phi(s)\|^2 + \|\phi(s)\|^2 + \|g(\phi(s))\|^2 + \|u(s)\|^2) ds,$$

then from (5.1.2), (5.2.16) and (5.2.17) we deduce that

$$\int_0^t \epsilon \|\phi_{tt}(s)\|^2 \le \frac{c}{\epsilon}(t+1), \quad \forall t \ge 0. \tag{5.2.18}$$

Also, from $(5.0.1)_2$ and (5.2.17), we deduce that

$$\begin{aligned} \int_0^t \|u_t(s)\|^2 ds &\le c \int_0^t (\|\phi_t(s)\|^2 + \|\Delta u(s)\|^2) ds \\ &\le c(t+1), \quad \forall t \ge 0. \end{aligned} \tag{5.2.19}$$

Finally, we have that,

$$\begin{aligned} &\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \\ &\le \|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{H}_{1,\epsilon}} + \|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \ \forall t, t' \in [t^*, 2t^*]. \end{aligned}$$

Indeed, on the one hand, from (5.2.18) and (5.2.19), we have

$$\begin{aligned} &\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{H}_{1,\epsilon}} \\ &\le c\left(\|\phi(t) - \phi(t')\|_1 + \sqrt{\epsilon}\|\phi_t(t) - \phi_t(t')\| + \|u(t) - u(t')\|\right) \\ &\le c \int_t^{t'} (\|\phi_t(s)\|_1 + \sqrt{\epsilon}\|\phi_{tt}(s)\| + \|u_t(s)\|) ds \\ &\le c(\epsilon, t^*)|t' - t|^{1/2}. \end{aligned}$$

On the other hand, it follows from (5.2.15) that

$$\|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \leq c(t^*)\|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \ \ \forall t' \geq 0. \tag{5.2.20}$$

Hence, we conclude with

$$\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \leq c(\epsilon, t^*)(|t' - t|^{1/2} + \|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}). \tag{5.2.21}$$

We check the assumptions of Theorem 1.2. Assumption 2 follows from estimates (5.2.4) and (5.2.5) of Proposition 5.1. Assumption 4 and 5 follows from (5.1.6) and (5.2.21) respectively. This shows the existence of exponential attractors $\mathcal{E}_\epsilon$ in $\mathcal{H}_1$. ∎

5.3 Robust family of exponential attractors

We start by showing the existence of an absorbing set in $\mathcal{H}_3$.

Proposition 5.2 *The semigroup $S_\epsilon(t)$ possesses an exponentially attracting bounded absorbing set $\mathcal{B}_3$ in $\mathcal{H}_3$.*

Proof. Let $B \subset \mathcal{H}_3$ be a bounded set, and let $(\phi_0, \phi_1, u_0) \in B$. Hence, since $\mathcal{H}_3 \subset \mathcal{H}_2$, then there exists $t(B) > 0$ such that $(\phi(t), \phi_t(t), u(t)) \in \mathcal{B}_2, \quad \forall t \geq t(B)$. That is,

$$\|\phi(t)\|_2^2 + \epsilon\|\phi_t(t)\|_1^2 + \|u(t)\|_1^2 \leq r_2, \quad \forall t \geq t(B). \tag{5.3.1}$$

The following estimates hold true.

$$
\begin{aligned}
(\Delta g(\phi), \Delta\phi_t) &\le \|g'(\phi)\|_{L^\infty(\Omega)}\|\Delta\phi\|\|\Delta\phi_t\| + \|g''(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^4\Omega}\|\Delta\phi_t\| \\
&\le c(\|g'(\phi)\|^2_{L^\infty(\Omega)}\|\Delta\phi\|^2 + \|g''(\phi)\|^2_{L^\infty(\Omega)}\|\nabla\phi\|_1^4) + \frac{1}{2}\|\Delta\phi_t\|^2,
\end{aligned}
\tag{5.3.2}
$$

$$
\begin{aligned}
(g(\phi), \Delta^2\phi) &\le \|g'(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|\|\nabla\Delta\phi\| \\
&\le \|g'(\phi)\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2 + \frac{1}{4}\|\nabla\Delta\phi\|^2, && (5.3.3)\\
(u, \Delta^2\phi) &\le \|\nabla u\|^2 + \frac{1}{4}\|\nabla\Delta\phi\|^2, && (5.3.4)\\
\epsilon(\Delta\phi, \Delta\phi_t) &\le \frac{1}{2}\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2. && (5.3.5)
\end{aligned}
$$

Multiply $(5.0.1)_1$ by $\Delta^2\phi_t$ and $\kappa\Delta^2\phi$, with $0 < \kappa \le \frac{1}{8}$, then multiply $(5.0.1)_2$ by $\Delta^2 u$, and we integrate over Ω. Addition of the resulting equations gives, on account of (5.3.2)-(5.3.5),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\nabla\Delta\phi\|^2 + (1+\kappa)\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2 + 2\kappa\epsilon(\Delta\phi, \Delta\phi_t)\right] \\
&+ \frac{\kappa}{2}\|\nabla\Delta\phi|^2 + \frac{\kappa}{2}\|\Delta\phi\|^2 + \epsilon\left(\frac{1}{2} - 2\kappa\right)\|\Delta\phi_t\|^2 + \epsilon\kappa(\Delta\phi, \Delta\phi_t) \\
&\le c(\|g'(\phi)\|^2_{L^\infty(\Omega)}\|\Delta\phi\|^2 + \|g''(\phi)\|^2_{L^\infty(\Omega)}\|\nabla\phi\|_1^4 + \|\nabla u\|^2).
\end{aligned}
$$

Hence from (5.3.1), there exists a constant $\varpi_1 > 0$ independent of ϵ such that

$$
\frac{d}{dt}E_3(t) + \varpi_1 E_3(t) \le c(r_2), \tag{5.3.6}
$$

where

$$
E_3(t) = \|\nabla\Delta\phi\|^2 + (1+\varpi)\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2 + 2\varpi\epsilon(\Delta\phi, \Delta\phi_t).
$$

Clearly, by Hölder's and Young's inequality, there exists constants $\varpi_2, \varpi_3 > 0$, independent of ϵ such that

$$\varpi_2(\|\nabla\Delta\phi\|^2 + \|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2) \leq E_3(t)$$

$$\leq \varpi_3(\|\nabla\Delta\phi\|^2 + \|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2). \tag{5.3.7}$$

Application of the generalized Gronwall's lemma to (5.3.6) and due to (5.3.7), we obtain

$$\|(\phi(t), \phi_t(t), u(t))\|^2_{\mathcal{H}_{3,\epsilon}} \leq c(B)e^{-\varpi_1 t} + c(r_2), \forall t \geq 0, \tag{5.3.8}$$

Hence, we have that

$$\mathcal{B}_3 = \{(\varphi, \psi, v) \in \mathcal{H}_3, \|(\varphi, \psi, v)\|_{\mathcal{H}_{3,\epsilon}} \leq \sqrt{2c(r_2)/\varpi_1} = r_3\}$$

is an exponentially attracting absorbing set for $S_\epsilon(t)$ on $\mathcal{H}_3$. ∎

We prove the following result.

Proposition 5.3 *For every $\epsilon \in (0,1]$, there exists $c > 0$, independent of ϵ, such that for any* $\mathrm{z} \in \mathcal{B}_3$,

$$\|S_\epsilon(t)\mathrm{z}\|_{\mathcal{H}_{2,0}} \leq c, \quad \forall t \geq 1. \tag{5.3.9}$$

Proof. Let $\mathrm{z}_0 = (\phi_0, \phi_1, u_0) \in \mathcal{B}_3$. We set $(\phi(t), \phi_t(t), u(t)) = S_\epsilon(t)(\phi_0, \phi_1, u_0)$, $\forall t \geq 0$. There exists $c > 0$, independent of ϵ, such that

$$\|\phi(t)\|_3^2 + \epsilon\|\phi_t(t)\|_2^2 + \|u(t)\|_2^2 \leq c, \quad \forall t \geq 0. \tag{5.3.10}$$

Multiplying the first equation of (5.0.1) by $\Gamma\phi_t$, where $\Gamma = I - \Delta$, then integrating over Ω, we obtain

$$\frac{\epsilon}{2}\frac{d}{dt}\|\phi_t\|_1^2 + \|\phi_t\|_1^2 + (-\Delta\phi, \Gamma\phi_t) + (\phi, \Gamma\phi_t) + (g(\phi), \Gamma\phi_t) - (u, \Gamma\phi_t) = 0.$$

Hence, we deduce due to (5.3.10), that

$$\epsilon \frac{d}{dt}\|\phi_t\|_1^2 + \|\phi_t\|_1^2 \leq c. \tag{5.3.11}$$

Then proceeding like in the proof of Proposition 2.5, we get that

$$\|\phi_t(t)\|_1^2 \leq c, \quad \forall t \geq 1. \tag{5.3.12}$$

Hence estimate (5.3.9). ∎

The following estimate holds for differences of two solutions.

Proposition 5.4 *There exist $t_\star > 0$, c and $c' > 0$ all independent of ϵ such that*

$$\|S_\epsilon(t)(\phi_0, \phi_1, u_0) - \mathcal{L}S(t)(\phi_0, u_0)\|_{\mathcal{H}_{1,\epsilon}}^2 \leq c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \geq t_\star, \tag{5.3.13}$$

for any $(\phi_0, \phi_1, u_0) \in \mathcal{B}_3$, and

$$\|S_\epsilon(t)(\phi_0, \phi_1, u_0) - \mathcal{L}S(t)(\phi_0, u_0)\|_{\mathcal{H}_{1,0}}^2 \leq c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \geq t_\star, \tag{5.3.14}$$

for any $(\phi_0, \phi_1, u_0) \in S_\epsilon(1)\mathcal{B}_3$, and any $\epsilon \in (0, 1]$, where $\mathcal{L}(\psi(t), \upsilon(t)) = (\psi(t), \mathscr{L}(\psi(t), \upsilon(t)), \upsilon(t))$.

Proof. Let $(\phi_0, \phi_1, u_0) \in \mathcal{B}_3$. We set

$(\phi^\epsilon(t), \phi_t^\epsilon(t), u^\epsilon(t)) = S_\epsilon(t)(\phi_0, \phi_1, u_0)$, and $(\phi(t), \phi_t(t), u(t)) = \mathcal{L}S(t)(\phi_0, u_0)$.

We have that

$$\|\phi^\epsilon(t)\|_3^2 + \epsilon\|\phi_t^\epsilon(t)\|_2^2 + \|u^\epsilon(t)\|_2^2 \leq c, \quad \forall t \geq 0, \tag{5.3.15}$$

$$\|\phi(t)\|_3^2 + \|u(t)\|_2^2 \leq c, \quad \forall t \geq 0. \tag{5.3.16}$$

We set $P = \phi^\epsilon - \phi$ and $R = u^\epsilon - u$, then the pair (P, R) satisfies the problem:

$$\begin{cases} \epsilon P_{tt} + P_t - \Delta P + P + g(\phi^\epsilon) - g(\phi) - R = -\epsilon \phi_{tt}, \\ R_t + P_t - \Delta R = 0, \\ P|_{t=0} = 0, \quad P_t|_{t=0} = \phi_1 - \mathscr{L}(\phi_0, u_0), \quad R|_{t=0} = 0. \end{cases} \tag{5.3.17}$$

Using the same method as in the proof of Proposition 2.6, we reach the estimate

$$\|P(t)\|_1^2 + \epsilon\|P_t(t)\|^2 + \|R(t)\|^2 \le c\sqrt[4]{\epsilon}(\epsilon + \epsilon\|\phi_1 - \mathscr{L}(\phi_0, u_0)\|_1^2)e^{c't}, \quad \forall t > \sqrt{\epsilon}. \tag{5.3.18}$$

Finally, estimate (5.3.13) follows from (5.3.9) while estimate (5.3.14) follows from (5.3.9) and (5.3.18). ∎

We have the following Corollary of Proposition 5.4.

Corollary 5.3.1 .

$$\|\Pi_\epsilon S_\epsilon(t)(\phi_0, \phi_1, u_0) - S(t)(\phi_0, u_0)\|_{\mathcal{H}_{1,0}}^2 \le c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \ge t_\star, \tag{5.3.19}$$

where $\Pi_\epsilon(X \times Y \times Z) = X \times Z$, *i.e.,*

$\|\phi^\epsilon(t) - \phi(t)\|_1^2 + \|u^\epsilon(t) - u(t)\|^2 \le c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \ge t_\star.$

The semigroup $S(t)$ for the variable (ϕ, u) alone possesses an exponential attractor $\mathcal{E}_0$ on $\mathcal{H}_{1,0}$, see Theorem 9.14 in [45]. We set $\widetilde{\mathcal{B}}_3 = S_\epsilon(t^*)\mathcal{B}_3$, where $t^* > 0$ is independent of ϵ.

Theorem 5.3 *There exist* $\varpi_1, \varpi_2 \in (0, \frac{1}{2}]$ *and* $M_1, M_2 > 0$, *all independent of* ϵ *and a family of exponential attractors* $\mathcal{E}_\epsilon$ *enjoying all the properties of Theorem 5.2 and such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_{1,\epsilon}}(\mathcal{E}_\epsilon, \mathcal{E}) \le M_1 \epsilon^{\varpi_1}, \tag{5.3.20}$$

$$\mathrm{dist}_{\mathcal{H}_{1,0}}(\mathcal{E}_\epsilon, \mathcal{E}) \le M_2 \varepsilon^{\varpi_2}, \tag{5.3.21}$$

$$\text{and} \quad \lim_{\epsilon \to 0} \mathrm{dist}_{\mathcal{H}_{1,0}}(\mathcal{E}, \mathcal{E}_\epsilon) = 0, \tag{5.3.22}$$

where $\mathcal{E} = \{(\varphi, \mathscr{L}(\varphi, \vartheta), \vartheta),\, (\varphi, \vartheta) \in \mathcal{E}_0\}$.

Proof. On account of Theorem 1.2, we let $E_\epsilon = \mathcal{U}_1$, $V_\epsilon = \mathcal{U}_2$, $W_\epsilon = \mathcal{U}_3$, $B_\epsilon = \widetilde{\mathcal{B}}_4$ and we check all the assumptions 1-5. To verify Assumption 1, using the interpolation inequality, there exists a constant c such that for some $\theta \in [0, 1]$ we have

$$\begin{aligned} \|\mathscr{L}(\varphi, \vartheta) - \mathscr{L}(\psi, v)\| &\le \|N(\varphi - \psi)\| + \|\varphi - \psi\| + \|g(\varphi) - g(\psi)\| + \|\vartheta - v\| \\ &\le c\big(\|\varphi - \psi\|^{1/2} + \|\varphi - \psi\|_3^{1/2}\big)\|\varphi - \psi\|_1^{1/2} + \|\vartheta - v\| \\ &\le c(\|\varphi - \psi\|_1^{1/2} + \|\vartheta - v\|^{1/2}), \end{aligned} \tag{5.3.23}$$

for any (φ, ϑ) and (ψ, v) in $\mathcal{B}$.

Assumption 2, 4 and 5 were proven in Theorem 5.2. Assumption 3 follows from (5.3.13) and (5.3.14). This shows the existence of exponential attractors in $\mathcal{H}_1$ that satisfy (5.3.20), (5.3.21) and (5.3.22). ∎

We also state the following Theorem which is a direct consequence of Corollary 5.3.19.

Theorem 5.4 *For every* $\epsilon \in (0, 1]$, *there exists a constant* $M_1 > 0$ *independent of* ϵ *such that the family of exponential attractors* $\mathcal{E}_\epsilon$ *of the semigroup* $S_\epsilon(t)$ *on* $\mathcal{H}_1$ *satisfies*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_{1,0}}(\Pi_\epsilon \mathcal{E}_\epsilon, \mathcal{E}_0) \leq M_1 \sqrt[4]{\epsilon}. \tag{5.3.24}$$

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